Neutrino oscillations in quantum mechanics and quantum field theory

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**But:** A closer look reveals a number of subtle and even paradoxical issues

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- When are the oscillations described by a universal probability?
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- Do Mössbauer neutrinos oscillate?
Debating the basics of neutrino oscillations ... 


Clarification of some of these issues and some apparent paradoxes of neutrino oscillations in:

Neutrino flavour mixing and oscillations

Diagonalization of the mass terms of the charged leptons and neutrinos gives

$$\Delta L = -\frac{g}{\sqrt{2}} (\bar{e}_\alpha \gamma^\mu U_{\alpha i} \nu_{iL}) W^-_\mu + \text{diag. mass terms} + h.c.$$ 

$$\alpha = e, \mu, \tau, \quad i = 1, 2, 3$$

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i \quad \Rightarrow \quad |\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle$$

The standard formula for the oscillation probability of relativistic neutrinos in vacuum:

$$P(\nu_\alpha \rightarrow \nu_\beta; L) = \left| \sum_i U_{\beta i} e^{-i \frac{\Delta m^2_{ij}}{2p} L} U^*_{\alpha i} \right|^2$$
How is it usually derived?

Assume at time \( t = 0 \) and coordinate \( x = 0 \) a flavour eigenstate \(|\nu_a\rangle\) is produced:

\[
|\nu(0, 0)\rangle = |\nu^\text{fl}_\alpha\rangle = \sum_i U^*_\alpha i |\nu^\text{mass}_i\rangle
\]

After time \( t \) at the position \( x \), for plane-wave particles:

\[
|\nu(t, x)\rangle = \sum_i U^*_\alpha i e^{-ip_i x} |\nu^\text{mass}_i\rangle
\]

Mass eigenstates pick up the phase factors \( e^{-i\phi_i} \) with

\[
\phi_i \equiv p_i x = Et - \vec{p} \cdot \vec{L}
\]

\[
P(\nu_\alpha \to \nu_\beta) = \left| \langle \nu^\text{fl}_\beta | \nu(t, \vec{L}) \rangle \right|^2
\]
How is it usually derived?

Consider $\vec{p}||\vec{L} \Rightarrow \vec{p}\vec{L} = pL \quad (p = |\vec{p}|, \ L = |\vec{L}|)$

Phase differences between different mass eigenstates:

$$\Delta \phi = \Delta E \cdot t - \Delta p \cdot L$$

Shortcuts to the standard formula

1. Assume the emitted neutrino state has a well defined momentum (same momentum prescription) $\Rightarrow \Delta p = 0$.

For ultra-relativistic neutrinos $E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \Rightarrow$

$$\Delta E \approx \frac{m_2^2 - m_1^2}{2E} \equiv \frac{\Delta m^2}{2E}; \quad t \approx L \quad (\hbar = c = 1)$$

$\Rightarrow$ The standard formula is obtained
How is it usually derived?

2. Assume the emitted neutrino state has a well defined energy (same energy prescription) \( \Rightarrow \Delta E = 0 \).

\[ \Delta \phi = \Delta E \cdot t - \Delta p \cdot L \Rightarrow - \Delta p \cdot L \]

For ultra-relativistic neutrinos

\[ p_i = \sqrt{E^2 - m_i^2} \approx E - \frac{m_i^2}{2p} \Rightarrow \]

\[ -\Delta p \equiv p_1 - p_2 \approx \frac{\Delta m^2}{2E} ; \]

\[ \Rightarrow \text{The standard formula is obtained} \]

Stand. phase \( \Rightarrow \)

\[ (l_{osc})_{ik} = \frac{4\pi E}{\Delta m^2_{ik}} \approx 2.5 \ m \frac{E \text{ (MeV)}}{\Delta m^2_{ik} \text{ eV}^2} \]
Same $E$ and same $p$ approaches
Very simple and transparent
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Allow one to quickly arrive at the desired result
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Very simple and transparent

Allow one to quickly arrive at the desired result

Trouble: they are both wrong
Kinematic constraints

Same momentum and same energy assumptions: contradict kinematics!

Pion decay at rest \((\pi^+ \rightarrow \mu^+ + \nu_\mu, \quad \pi^- \rightarrow \mu^- + \bar{\nu}_\mu)\):

For decay with emission of a massive neutrino of mass \(m_i\):

\[
E_i^2 = \frac{m_{\pi}^2}{4} \left( 1 - \frac{m_{\mu}^2}{m_{\pi}^2} \right)^2 + \frac{m_i^2}{2} \left( 1 - \frac{m_{\mu}^2}{m_{\pi}^2} \right) + \frac{m_i^4}{4m_{\pi}^2}
\]

\[
p_i^2 = \frac{m_{\pi}^2}{4} \left( 1 - \frac{m_{\mu}^2}{m_{\pi}^2} \right)^2 - \frac{m_i^2}{2} \left( 1 + \frac{m_{\mu}^2}{m_{\pi}^2} \right) + \frac{m_i^4}{4m_{\pi}^2}
\]

For massless neutrinos: \(E_i = p_i = E \equiv \frac{m_{\pi}^2}{2} \left( 1 - \frac{m_{\mu}^2}{m_{\pi}^2} \right) \approx 30\text{ MeV}\)

To first order in \(m_i^2\):

\[
E_i \approx E + \xi \frac{m_i^2}{2E}, \quad p_i \approx E - (1 - \xi) \frac{m_i^2}{2E}, \quad \xi = \frac{1}{2} \left( 1 - \frac{m_{\mu}^2}{m_{\pi}^2} \right) \approx 0.2
\]
How can two different and wrong assumptions lead to the same (correct) answer?
When are neutrino oscillations observable?

Keyword: Coherence

Neutrino flavour eigenstates $\nu_e$, $\nu_\mu$ and $\nu_\tau$ are coherent superpositions of mass eigenstates $\nu_1$, $\nu_2$ and $\nu_3 \Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate $E$ and $p$ measurements one can tell (through $E = \sqrt{p^2 + m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear! $\Rightarrow$ Coherent production/detection conditions $\Delta E \ll \sigma_E$, $\Delta p \ll \sigma_P$. Equivalent to localization conditions: $L_S, L_D \ll l_{osc}$.

Coherent propagation: no wave packet separation due to $\Delta v \neq 0 \Rightarrow$

$$L \ll l_{coh} = \frac{v}{\Delta v} \sigma_x$$
Oscillations and QM uncertainty relations

Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations.

Neutrino energy and momentum are characterized by uncertainties $\sigma_E$ and $\sigma_p$, related to the spatial localization and time scale of the production and detection processes. These uncertainties:

- allow the emitted/absorbed neutrino state to be a coherent superposition of different mass eigenstates (Kayser, 1981)
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation (Nussinov, 1976)

$\sigma_E$ – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for $\sigma_p$. 
What determines the length of $\nu$ w. packets?

Energy and momentum uncertainties $\Rightarrow$ neutrinos are described by wave packets rather than by plane waves.

$$\sigma_x \simeq \frac{1}{\sigma_{p\text{ eff}}}. $$

The effective neutrino momentum uncertainty:

$$\frac{1}{\sigma_{p\text{ eff}}^2} = \frac{1}{\sigma_p^2} + \frac{(\vec{v}_g - \vec{v}_P)^2}{\sigma_E^2} $$

$\sigma_E \leq \sigma_p \quad \Leftarrow \quad$ for $\nu'$s produced both in decays and collisions.

$$\sigma_{p\text{ eff}} \simeq \frac{\sigma_E}{v_g}, $$

$$\sigma_x \simeq \frac{v_g}{\sigma_E}.$$
Plane wave approach: plagued with inconsistencies. If applied correctly, does not lead to neutrino oscillations at all!
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Consistent approaches:

- QM wave packet approach – neutrinos described by wave packets rather than by plane waves
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Consistent approaches:

- QM wave packet approach – neutrinos described by wave packets rather than by plane waves
- QFT approach: neutrino production and detection explicitly taken into account. Neutrinos are intermediate particles described by propagators
QM wave packet approach

The evolved produced state:

$$|\nu^f_\alpha(x, t)\rangle = \sum_i U^*_{\alpha i} |\nu_{\text{mass}}^i(x, t)\rangle = \sum_i U^*_{\alpha i} \Psi^P_i(x, t)|\nu_{\text{mass}}^i\rangle$$

The coordinate-space wave function of the $i$th mass eigenstate (w. packet):

$$\Psi^P_i(x, t) = \int \frac{d^3 p}{(2\pi)^3} f^S_i(p) e^{i\frac{p}{\hbar} - iE_i(p)t}$$

Momentum distribution function $f^S_i(p)$: sharp maximum at $p = P$ (width of the peak $\sigma_{PP} \ll P$).

$$E_i(p) = E_i(P) + \frac{\partial E_i(p)}{\partial p} \bigg|_{p=P} (p - P) + \frac{1}{2} \frac{\partial^2 E_i(p)}{\partial p^2} \bigg|_{p_0} (p - P)^2 + \ldots$$

$$\vec{v}_i = \frac{\partial E_i(p)}{\partial p} = \frac{\vec{p}}{E_i}, \quad \alpha = \frac{\partial^2 E_i(p)}{\partial p^2} = \frac{m_i^2}{E_i^2}.$$
Evolved neutrino state

\[ \Psi_i^P(\vec{x}, t) \simeq e^{-iE_i(P)t + i\vec{P}\vec{x}} g_i^P(\vec{x} - \vec{v}_i t) \quad (\alpha \to 0) \]

\[ g_i^P(\vec{x} - \vec{v}_i t) \equiv \int \frac{d^3p_1}{(2\pi)^3} f_i^P(\vec{p}_1) e^{i\vec{p}_1(\vec{x} - \vec{v}_g t)} \]

Center of the wave packet: \( \vec{x} - \vec{v}_i t = 0 \). Spatial length: \( \sigma_{xP} \sim 1/\sigma_{pP} \) (\( g_i^S \) decreases quickly for \( |\vec{x} - \vec{v}_i t| \geq \sigma_{xP} \)).

Detected state (centered at \( \vec{x} = \vec{L} \)):

\[ |\nu^f_\beta(\vec{x})\rangle = \sum_k U_{\beta k}^* \Psi_k^D(\vec{x}) |\nu^\text{mass}_i\rangle \]

The coordinate-space wave function of the \( i \)th mass eigenstate (w. packet):

\[ \Psi_k^D(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} f_k^D(\vec{p}) e^{i\vec{p}(\vec{x} - \vec{L})} \]
Oscillation probability

Transition amplitude:

\[ \mathcal{A}_{\alpha\beta}(T, \vec{L}) = \langle \nu_\beta^f | \nu_\alpha^f(T) \rangle = \sum_i U_{\alpha i}^* U_{\beta i} \mathcal{A}_i(T, \vec{L}) \]

\[ \mathcal{A}_i(T, \vec{L}) = \int \frac{d^3p}{(2\pi)^3} f_i^P(\vec{p}) f_i^{D*}(\vec{p}) e^{-iE_i(p)T+i\vec{p}\vec{L}} \]

Strongly suppressed unless \( |\vec{L} - \vec{v}_i T| \lesssim \sigma_x \). E.g., for Gaussian wave packets:

\[ \mathcal{A}_i(T, \vec{L}) \propto \exp \left[ -\frac{(\vec{L} - \vec{v}_i T)^2}{4\sigma_x^2} \right] , \quad \sigma_x^2 \equiv \sigma_{xP}^2 + \sigma_{xD}^2 \]

Oscillation probability:

\[ P(\nu_\alpha \rightarrow \nu_\beta; T, \vec{L}) = |\mathcal{A}_{\alpha\beta}|^2 = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^* \mathcal{A}_i(T, \vec{L}) \mathcal{A}_k^*(T, \vec{L}) \]
Oscillation phase

Oscillations are due to phase differences of different mass eigenstates:

\[ \Delta \phi = \Delta E \cdot t - \Delta p \cdot L \quad (E_i = \sqrt{p_i^2 + m_i^2}) \]

Consider the case \( \Delta E \ll E \) (relativistic or quasi-degenerate neutrinos) \( \Rightarrow \)

\[ \Delta E = \frac{\partial E}{\partial p} \Delta p + \frac{\partial E}{\partial m^2} \Delta m^2 = v \Delta p + \frac{1}{2E} \Delta m^2 \]

\[ \Delta \phi = (v \Delta p + \frac{1}{2E} \Delta m^2) \cdot t - \Delta p \cdot L \]

\[ = - (L - vt) \Delta p + \frac{\Delta m^2}{2E} \cdot t \]

In the center of wave packet \( (L - vt) = 0 \). In general, \( |L - vt| \lesssim \sigma_x \);

if \( \sigma_x \Delta p \ll 1 \) (i.e. \( \Delta p \ll \sigma_p \)), \( |L - vt| \Delta p \ll 1 \) \( \Rightarrow \)
$$\Delta \phi = \frac{\Delta m^2}{2E} t,$$

$$L \simeq vt \simeq t$$

- the result of the “same momentum” approach recovered!
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\[ \diamond \quad \Delta \phi = -\frac{1}{v}(L - vt)\Delta E + \frac{\Delta m^2}{2p}L \quad \Rightarrow \quad \frac{\Delta m^2}{2p}L \]
\[ \Delta \phi = \frac{\Delta m^2}{2E} t, \quad L \simeq vt \simeq t \]

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– for \( \Delta E \sigma_x/v \ll 1 \) (i.e. \( \Delta E \ll \sigma_E \)) – “same energy” result recovered.
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The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with \( \Delta E \ll E \)
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– for \( \Delta E \sigma_x / v \ll 1 \) (i.e. \( \Delta E \ll \sigma_E \)) – “same energy” result recovered.

The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with \( \Delta E \ll E \)
- Neutrino energy uncertainty \( \sigma_E \gg \Delta E \) (typically this means \( \sigma_x \ll l_{osc} \))
Oscillation probability

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments ⇒ integration over $T$:

$$P(\nu_\alpha \rightarrow \nu_\beta; L) = \int dT \ P(\nu_\alpha \rightarrow \nu_\beta; T, L) = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^* e^{-i\Delta m_{ik}^2 L / 2P} F_{ik}$$
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$$F_{ik} = \int \frac{dq}{2\pi v} f^P_i (r_{ik} q - \Delta E_{ik}/2v + P_i) f^{D*}_i (r_{ik} q - \Delta E_{ik}/2v + P_i)$$

$$\times f_P^D (r_{ik} q + \Delta E_{ik}/2v + P_k) f^D_k (r_{ik} q + \Delta E_{ik}/2v + P_k) e^{i \frac{\Delta v}{v} qL}$$

Here:

$$v \equiv \frac{v_i + v_k}{2}, \quad \Delta v \equiv v_k - v_i, \quad r_{i,k} \equiv \frac{v_{i,k}}{v}$$
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$$F_{ik} = \int \frac{dq}{2\pi v} f_i^P (r_k q - \Delta E_{ik} / 2v + P_i) f_i^{D*} (r_k q - \Delta E_{ik} / 2v + P_i) \times f_k^{P*} (r_i q + \Delta E_{ik} / 2v + P_k) f_k^D (r_i q + \Delta E_{ik} / 2v + P_k) e^{i \frac{\Delta v}{v} qL}$$

Here:

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For $(\Delta v/v) \sigma_p L \ll 1$ (i.e. $L \ll l_{coh} = (v/\Delta v) \sigma_x$) $F$ is approximately independent of $L$; in the opposite case $F$ is strongly suppressed
Oscillation probability

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments \( \Rightarrow \) integration over \( T \):

\[
P(\nu_\alpha \rightarrow \nu_\beta; L) = \int dT \, P(\nu_\alpha \rightarrow \nu_\beta; T, L) = \sum_{i,k} U^*_{\alpha i} U_{\beta i} U_{\alpha k}^* U_{\beta k} e^{-i \frac{\Delta m^2_{i,k}}{2P} L} F_{ik}
\]

\[
F_{ik} = \int \frac{dq}{2\pi v} f^P_i (r_k q - \Delta E_{ik}/2v + P_i) f^{D*}_i (r_k q - \Delta E_{ik}/2v + P_i)
\]

\[
\times f^P_k (r_i q + \Delta E_{ik}/2v + P_k) f^D_k (r_i q + \Delta E_{ik}/2v + P_k) e^{i \frac{\Delta v}{v} qL}
\]

Here:

\[
v \equiv \frac{v_i + v_k}{2}, \quad \Delta v \equiv v_k - v_i, \quad r_{i,k} \equiv \frac{v_{i,k}}{v}
\]

- For \( (\Delta v/v)\sigma_p L \ll 1 \) (i.e. \( L \ll l_{coh} = (v/\Delta v)\sigma_x \)) \( F \) is approximately independent of \( L \); in the opposite case \( F \) is strongly suppressed

- \( F \) is also strongly suppressed unless \( \Delta E_{ik}/v \ll \sigma_p \), i.e. \( \Delta E_{ik} \ll \sigma_E \)
  - coherent production/detection condition
Normalisation prescription

Oscillation probability calculated in QM w. packet approach is not automatically normalized! Can be normalized “by hand” by imposing the unitarity condition:

\[ \sum_{\beta} P_{\alpha\beta}(L) = 1. \]

This gives

\[ F_{ii} = \int \frac{dp}{2\pi v} |f^P_i(p)|^2 |f^D_i(p)|^2 = 1 \]

– important for proving Lorentz invariance of the oscillation probability.

Depends on the overlap of \( f^P_i(p) \) and \( f^D_i(p) \) \( \Rightarrow \) no independent normalization of the produced and detected neutrino wave function would do!

In QFT approach the correctly normalized \( P_{\alpha\beta}(L) \) is automatically obtained and the meaning of the normalization procedure adopted in the w. packet approach clarified.
Shortcomings of the QM w. packet approach

- Neutrino wave packet postulated rather than derived, widths estimated
- Production and detection processes are not considered
- Inadequate normalization procedure. Normalization “by hand” is unavoidable.

Advantage: simplicity
Calc. from 1st principles – QFT approach

Production - propagation - detection treated as a single inseparable process. External particles are described by wave packets, neutrinos – by propagators.

One-particle states of external particles:

\[ |A\rangle = \int [dp] f_A(\vec{P}, \vec{p}) |A, \vec{p}\rangle , \quad [dp] \equiv \frac{d^3p}{(2\pi)^3 \sqrt{2E_A(\vec{p})}} \]

\[ |A, \vec{p}\rangle \text{ – one-particle momentum eigenstate corresponding to momentum } \vec{p} \]

and energy \( E_A(\vec{p}) \) (free particles: \( E_A(\vec{p}) = \sqrt{\vec{p}^2 + m_A^2} \)).

\( f_A(\vec{p}, \vec{P}) \) – momentum distribution function with the mean momentum \( \vec{P} \).

Normalization condition: \( \langle A|A\rangle = 1 \Rightarrow \int d^3p |f_A(\vec{p})|^2 / (2\pi)^3 = 1 \).

Coordinate-space wave packet with maximum at \( \vec{x} = \vec{x}_0 \) at the time \( t - t_0 \):

\[ \Psi_A(x) = \int \frac{d^3p}{(2\pi)^3} f_A(\vec{p}) e^{-iE_A(\vec{p})(t-t_0) + i\vec{p}(\vec{x}-\vec{x}_0)} \]
QFT approach – contd.

\[ P_i(q) \quad \longrightarrow \quad \nu \quad \longrightarrow \quad P_f(k) \]

\[ D_i(q') \quad \longrightarrow \quad \nu \quad \longrightarrow \quad D_f(k') \]

\[ |P_i\rangle = \int [dq] f_{P_i}(\vec{q}, \vec{Q}) |P_i, \vec{q}\rangle, \quad |P_f\rangle = \int [dk] f_{P_f}(\vec{k}, \vec{K}) |P_f, \vec{k}\rangle, \]

\[ |D_i\rangle = \int [dq'] f_{D_i}(\vec{q'}, \vec{Q'}) |D_i, \vec{q'}\rangle, \quad |D_f\rangle = \int [dk'] f_{D_f}(\vec{k'}, \vec{K'}) |D_f, \vec{k'}\rangle. \]

The transition amplitude:

\[ iA_{\alpha\beta} = \langle P_f D_f | \hat{T} \exp \left[ -i \int d^4 x H_I(x) \right] - 1 |P_i D_i\rangle, \]
In the second order in weak interaction:

\[ i A_{\alpha \beta} = \sum_j U_{\alpha j}^* U_{\beta j} \int [dq] f_{P i}(\vec{q}, \vec{Q}) \int [dk] f_{P f}^*(\vec{k}, \vec{K}) \]

\[ \times \int [dq'] f_{D i}(\vec{q}', \vec{Q}') \int [dk'] f_{D f}^*(\vec{k}', \vec{K}') \ i A_{p.w., j}^{p.w.}(q, k; q', k') . \]

Plane-wave amplitude:

\[ i A_{j}^{p.w.}(q, k; q', k') = \int d^4 x_1 \int d^4 x_2 \tilde{M}_D(q', k') e^{-i(q' - k')(x_2 - x_D)} \]

\[ \times i \int \frac{d^4 p}{(2\pi)^4} \frac{p + m_j}{p^2 - m_j^2 + i\epsilon} e^{-i p(x_2 - x_1)} \tilde{M}_P(q, k) e^{-i(q - k)(x_1 - x_P)} \]

\[ \tilde{M}_jP, \tilde{M}_jD \] – production and detection amplitudes with neutrino spinors excluded. Full amplitudes:

\[ M_{jP}(q, k) \equiv \frac{\bar{u}_{jL}(p)}{\sqrt{2p_0}} \tilde{M}_P(q, k) , \quad M_{jD}(q', k') \equiv \tilde{M}_D(q', k') \frac{u_{jL}(p)}{\sqrt{2p_0}} \]
QFT approach – contd.

\[ iA_{\alpha\beta} = i \sum_j U^*_{\alpha j} U_{\beta j} \int \frac{d^4p}{(2\pi)^4} \Phi_{jP}(p^0, \vec{p}) \Phi_{jD}(p^0, \vec{p}) \frac{2p_0 e^{-ip^0 T + i\vec{p}\vec{L}}}{p^2 - m_j^2 + i\epsilon}. \]

\[ \Phi_{jP}(p^0, \vec{p}) = \int d^4x_1' e^{ipx_1'} \int [dq] \int [dk] f_{P_i}(\vec{q}, \vec{Q}) f^*_{P_f}(\vec{k}, \vec{K}) e^{-i(q-k)x_1'} M_{jP}(q, k) \]

\[ \Phi_{jD}(p^0, \vec{p}) = \int d^4x_2' e^{-ipx_2'} \int [dq'] \int [dk'] f_{D_i}(\vec{q}', \vec{Q}') f^*_{D_f}(\vec{k}', \vec{K}') e^{-i(q'-k')x_2'} M_{jD}(q', k') \]

For \( L, T \gg 1/p \) – fast oscillating factor in \( iA_{\alpha\beta} \) \( \Rightarrow \) main contribution to integral over \( p^0 \) from the pole at \( p^0 = E_j(\vec{p}) - i\epsilon \) (on-shell neutrinos).

\[ iA_{\alpha\beta} = \Theta(T) \sum_j U^*_{\alpha j} U_{\beta j} \int \frac{d^3p}{(2\pi)^3} \Phi_{jP}(E_j(\vec{p}), \vec{p}) \Phi_{jD}(E_j(\vec{p}), \vec{p}) e^{-iE_j(\vec{p})T + i\vec{p}\vec{L}} \]
Compare with $A_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f^P_j(p) = \Phi_{jp}(E_j(p), \vec{p}), \quad f^D_j(p) = \Phi_{jd}^*(E_j(p), \vec{p}) ,$$
Compare with $A_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f_j^P(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p}), \quad f_j^D(\vec{p}) = \Phi_{jD}^*(E_j(\vec{p}), \vec{p}),$$

Easy to understand: $\Phi_{jP}(E_j(\vec{p}), \vec{p})$ is the probability amplitude of $\nu_j$ production process in which $\nu_j$ is emitted with momentum $\vec{p}$.

$\Rightarrow \quad \Phi_{jP}$ is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet $f_j^P(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_j^P(\vec{p})$ and $f_j^D(\vec{p})$ are not “canonically” normalized.

Alternative approaches:
QFT approach – contd.

Compare with \( A_{ab}(T, \vec{L}) \) obtained in the QM w. packet approach: the two amplitudes coincide if

\[
  f_j^P(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p}), \quad f_j^D(\vec{p}) = \Phi_{jD}^*(E_j(\vec{p}), \vec{p}),
\]

Easy to understand: \( \Phi_{jP}(E_j(\vec{p}), \vec{p}) \) is the probability amplitude of \( \nu \) production process in which \( \nu_j \) is emitted with momentum \( \vec{p} \).

\[\Rightarrow\] \( \Phi_{jP} \) is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet \( f_j^P(\vec{p}) \). Similarly for neutrino detection.

N.B.: \( f_j^P(\vec{p}) \) and \( f_j^D(\vec{p}) \) are not “canonically” normalized.

Alternative approaches:

\[
|P_f\nu_j\rangle = (S - 1)|P_i\rangle, \quad |\nu_j\rangle = \langle P_f|P_f\nu_j\rangle
\]
QFT approach – contd.

Compare with \( A_{ab}(T, \vec{L}) \) obtained in the QM w. packet approach: the two amplitudes coincide if

\[
f_j^P(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p}) , \quad f_j^D(\vec{p}) = \Phi_{jD}^*(E_j(\vec{p}), \vec{p}) ,
\]

Easy to understand: \( \Phi_{jP}(E_j(\vec{p}), \vec{p}) \) is the probability amplitude of \( \nu \) production process in which \( \nu_j \) is emitted with momentum \( \vec{p} \)

\( \Rightarrow \) \( \Phi_{jP} \) is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet \( f_j^P(\vec{p}) \). Similarly for neutrino detection.

N.B.: \( f_j^P(\vec{p}) \) and \( f_j^D(\vec{p}) \) are not “canonically” normalized.

Alternative approaches:

\[
\begin{align*}
|P_f \nu_j \rangle &= (S - \mathbb{1})|P_i \rangle , \\
|\nu_j \rangle &= \langle P_f | P_f \nu_j \rangle
\end{align*}
\]

In coord. space: \( \psi_{\nu j} = \) convolution of the \( \nu \) source (prod. amplitude) and retarded propagator
Compare with $A_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f_J^P(\vec{p}) = \Phi_{JP}(E_J(\vec{p}), \vec{p}), \quad f_J^D(\vec{p}) = \Phi_{JD}^*(E_J(\vec{p}), \vec{p}),$$

Easy to understand: $\Phi_{JP}(E_J(\vec{p}), \vec{p})$ is the probability amplitude of $\nu$ production process in which $\nu_j$ is emitted with momentum $\vec{p}$

$\Rightarrow$ $\Phi_{JP}$ is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet $f_J^P(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_J^P(\vec{p})$ and $f_J^D(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

$|P_f \nu_j\rangle = (S - 1)|P_i\rangle$, $|\nu_j\rangle = \langle P_f|P_f \nu_j\rangle$

In coord. space: $\psi_{\nu j} = \text{convolution of the } \nu \text{ source (prod. amplitude)}$ and retarded propagator

All three approaches give the same results.
General properties of $\nu$ w. packets in QFT

\[ f_j^P (\vec{p}) \simeq M_{jP}(Q, K) \int d^4 x \, e^{iE_j(\vec{p}) t - i\vec{p} \cdot \vec{x}} \int [dq] \int [dk] f_{Pi}(\vec{q}, \vec{Q}) f^*_{Pf}(\vec{k}, \vec{K}) e^{-i(\vec{q} - \vec{k}) \cdot \vec{x}} \]

Integral over $\vec{x}$ gives $\sim \delta^{(3)}(\vec{q} - \vec{k} - \vec{p})$. Since $f_{Pi}(\vec{q}, \vec{Q})$, $f_{Pf}(\vec{k}, \vec{K})$ are sharply peaked at $\vec{Q}$ and $\vec{K}$ $\Rightarrow$ $f_j^P (\vec{p})$ is sharply peaked at

\[ \vec{P} \equiv \vec{Q} - \vec{K}. \]  

Width of the peak: $\sigma_{PP} \simeq \max\{\sigma_{Pi}, \sigma_{Pf}\}$

For external particles described by plane waves:

\[ f_j^P (\vec{p}) = \frac{M_{jP}(Q, K)}{\sqrt{2E_{Pi}V \cdot 2E_{Pf}V}} \delta^{(4)}(Q - K - P) \]

In general: $f_j^P (\vec{p}) \Rightarrow M_{jP}(Q, K) \times \text{("smeared $\delta$-functions") representing approx. conservation of mean energies and mean momenta.}
Matching QM & QFT expressions for ν w. p.

Example – Gaussian wave packets for external particles. QFT gives

\[ f_j^P(\vec{p}) \propto \frac{[M_jP(Q, K)]}{(\sigma_{eP}\sigma^3_{pP})} \exp \left[-g_P(E_j(\vec{p}), \vec{p})\right], \]

\[ g_P(E_j(\vec{p}), \vec{p}) = \frac{(\vec{p} - \vec{P})^2}{4\sigma^2_{pP}} + \frac{[E_j(\vec{p}) - E_P - \vec{v}_P(\vec{p} - \vec{P})]^2}{4\sigma^2_{eP}}. \]

Here

\[ \vec{P} \equiv \vec{Q} - \vec{K}, \quad E_P \equiv E_{Pi}(\vec{Q}) - E_{Pf}(\vec{K}), \]

\[ \sigma^2_{pP} = \sigma^2_{pPi} + \sigma^2_{pPf}, \quad \sigma_{xP}\sigma_{pP} = \frac{1}{2}, \]

\[ \vec{v}_P \equiv \sigma^2_{xP} \left( \frac{\vec{v}_{Pi}}{\sigma^2_{xPi}} + \frac{\vec{v}_{Pf}}{\sigma^2_{xPf}} \right), \quad \Sigma_P \equiv \sigma^2_{xP} \left( \frac{\vec{v}_{Pi}^2}{\sigma^2_{xPi}} + \frac{\vec{v}_{Pf}^2}{\sigma^2_{xPf}} \right), \]

\[ \sigma^2_{eP} = \sigma^2_{pP}(\Sigma_P - \vec{v}_P^2) \equiv \sigma^2_{pP} \lambda_P, \quad 0 \leq \lambda_P \leq 1. \]
Matching QM & QFT expressions for $\nu$ w. p.

Compare with Gaussian wave packet in QM approach:

$$f_j^P(\vec{p}, \vec{P}) = \left(\frac{2\pi}{\sigma_{PP}^2}\right)^{3/4} \exp \left[-\frac{(\vec{p} - \vec{P})^2}{4\sigma_{PP}^2}\right]$$

To match the QM and QFT expression: expand $E_j(\vec{p})$ around $\vec{p} = \vec{P}$ and subst. into $g_P(E_j(\vec{p}), \vec{p})$:

$$g_P(E_j(\vec{p}), \vec{p}) = (p - P)^k \alpha^{kl} (p - P)^l - \beta^k(p - P)^k + \gamma_j$$

$$\alpha^{kl} = \frac{1}{4\sigma_{eP}^2} \left[\lambda_P \delta^{kl} + (v_j - v_P)^k (v_j - v_P)^l + \frac{E_j - E_P}{E_j} (\delta^{kl} - v_j^k v_j^l)\right],$$

$$\beta^k = -\frac{1}{2\sigma_{eP}^2} (E_j - E_P)(v_j - v_P)^k,$$

$$\gamma_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}.$$

Try to represent $g_P(E_j(\vec{p}), \vec{p})$ in the form

$$g_P(E_j(\vec{p}), \vec{p}) = (p - P_{\text{eff}})^k \alpha^{kl} (p - P_{\text{eff}})^l + \tilde{\gamma}_j,$$

$P_{\text{eff}} \equiv \vec{P} + \vec{\delta}$.
Matching QM & QFT expressions for $\nu$ w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}, \quad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2} \frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}. $$

Diagonalization of $\alpha^{kl}$ gives ($OZ||(\vec{v}_j - \vec{v}_P)$):

$$\left(\sigma_{pP \text{ eff}}^x\right)^2 = \left(\sigma_{pP \text{ eff}}^y\right)^2 = \sigma_{pP}^2, \quad \frac{1}{\left(\sigma_{pP \text{ eff}}^z\right)^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2},$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if
Matching QM & QFT expressions for \( \nu \) w. p.

\[
\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}, \quad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2} \lambda_P \frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}.
\]

Diagonalization of \( \alpha^{kl} \) gives \( (OZ|| (\vec{v}_j - \vec{v}_P)) \):

\[
(\sigma_{pP \text{ eff}}^x)^2 = (\sigma_{pP \text{ eff}}^y)^2 = \sigma_{pP}^2, \quad \frac{1}{(\sigma_{pP \text{ eff}}^z)^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2},
\]

\( \Rightarrow \) QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced by (anisotropic) effective ones:
  
  \[-(\vec{p} - \vec{P})^2/(4\sigma_{pP}^2) \rightarrow -[(p^x - P_{\text{eff}}^x)^2/4(\sigma_{pP}^x)^2 + (p^y - P_{\text{eff}}^y)^2/4(\sigma_{pP}^y)^2 + (p^z - P_{\text{eff}}^z)^2/4(\sigma_{pP}^z)^2].\]
Matching QM & QFT expressions for $\nu$ w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}, \quad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma^2_{eP}} \frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}.$$ 

Diagonalization of $\alpha^{kl}$ gives $$(OZ\parallel(\vec{v}_j - \vec{v}_P)):\n
\begin{align*}
\left(\sigma^x_{pP\text{ eff}}\right)^2 &= \left(\sigma^y_{pP\text{ eff}}\right)^2 = \sigma^2_{pP}, \\
\frac{1}{\left(\sigma^z_{pP\text{ eff}}\right)^2} &= \frac{1}{\sigma^2_{pP}} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma^2_{eP}},
\end{align*}

\Rightarrow \quad \text{QM neutrino wave packets can match those obtained QFT if}

- Momentum uncertainties of the neutrino mass eigenstates are replaced by (anisotropic) effective ones:
  $$-(\vec{p} - \vec{P})^2 / (4\sigma^2_{pP}) \rightarrow -\left[\frac{(p^x - P^x_{\text{ eff}})^2}{4\sigma^2_{pP}} + \frac{(p^y - P^y_{\text{ eff}})^2}{4\sigma^2_{pP}} + \frac{(p^z - P^z_{\text{ eff}})^2}{4\sigma^2_{pP}}\right].$$

- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \rightarrow \vec{P}_{\text{ eff}} = \vec{P} + \vec{\delta}$. 
Matching QM & QFT expressions for $\nu$ w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}, \quad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma^2_{eP}} \frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}.$$ 

Diagonalization of $\alpha^{kl}$ gives ($OZ\parallel(\vec{v}_j - \vec{v}_P)$):

$$\left(\sigma^x_{pP\text{ eff}}\right)^2 = \left(\sigma^y_{pP\text{ eff}}\right)^2 = \sigma^2_{pP}, \quad \frac{1}{\left(\sigma^z_{pP\text{ eff}}\right)^2} = \frac{1}{\sigma^2_{pP}} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma^2_{eP}}.$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced by (anisotropic) effective ones: $$-(\vec{p} - \vec{P})^2 / (4\sigma^2_{pP}) \rightarrow -\left[\left(p^x - P^x_{\text{ eff}}\right)^2 / 4(\sigma^x_{pP})^2 + \left(p^y - P^y_{\text{ eff}}\right)^2 / 4(\sigma^y_{pP})^2 + \left(p^z - P^z_{\text{ eff}}\right)^2 / 4(\sigma^z_{pP})^2 \right].$$
- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \rightarrow \vec{P}_{\text{ eff}} = \vec{P} + \vec{\delta}$.
- The wave packet of each neutrino mass eigenstate gets an extra factor $N_j = \exp[-\tilde{\gamma}_j]$. 
Matching QM & QFT expressions for $\nu$ w. p.

If \[ |E_i - E_j| \ll \sigma_{eP} \Rightarrow \]

factors $N_j$ are the same for all $\nu$ mass eigenstates, can be included in common normalization factor. In the opposite case – coherence of different neutrino mass eigenstates is lost.

$\sigma_{eP} \leq \sigma_{pP} \Rightarrow$ except for $\vec{v}_j \approx \vec{v}_P$ momentum uncertainty along $(\vec{v}_j - \vec{v}_P)$ is dominated by $\sigma_{eP}$.

In the stationary neutrino source limit $(\sigma_{eP}, \vec{v}_P \to 0)$, effective longitudinal mom. uncertainty $\sigma_{pP}^{zP_{\text{eff}}} = 0$ even though the true mom. uncertainty $\sigma_{pP} \neq 0$.

\[ \downarrow \]

Coherence length $l_{\text{coh}} \to \infty$
Oscillation probability in QFT

What is calculated in QFT is the probability of the overall production-propagation-detection process. How to extract from it the oscillation probability \( P_{\alpha\beta}(L) \)?

1. Recall the operational definition of \( P_{\alpha\beta}(L) \). Detection rate for \( \nu_\beta \):

\[
\Gamma_{\beta}^{\text{det}} = \int dE \ j_{\beta}(E) \sigma_\beta(E),
\]

If a source at a distance \( L \) from the detector emits \( \nu_\alpha \) with the energy spectrum \( d\Gamma_\alpha^{\text{prod}}(E)/dE \):

\[
J_{\beta}(E) = \frac{1}{4\pi L^2} \frac{d\Gamma_\alpha^{\text{prod}}(E)}{dE} P_{\alpha\beta}(L,E),
\]

\( \Rightarrow \) substitute into \( \Gamma_{\beta}^{\text{det}} \):
Oscillation probability in QFT

\[ \Gamma_{\alpha\beta}^{\text{tot}} \equiv \int dE \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)}{dE} = \frac{1}{4\pi L^2} \int dE \frac{d\Gamma_{\alpha}^{\text{prod}}(E)}{dE} P_{\alpha\beta}(L, E) \sigma_{\beta}(E) \]

\[ P_{\alpha\beta}(L, E) = \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)/dE}{\frac{1}{4\pi L^2} [d\Gamma_{\alpha}^{\text{prod}}(E)/dE] \sigma_{\beta}(E)}. \]

An important ingredient: the assumption that the overall rate factorizes into the production rate, propagation (oscillation) probability and detection cross section.

If this does not hold, oscillation probability is undefined \[ \Rightarrow \]

Need to deal instead with the overall rate of neutrino production, propagation and detection.
Oscillation probability in QFT

Try to cast \( P_{\alpha\beta}^{\text{tot}} \) in the same form (check if the factorization condition holds!)

\[
iA_{\alpha\beta} = i \sum_j U_{\alpha j}^* U_{\beta j} \int \frac{d^4 p}{(2\pi)^4} \Phi_j P(p^0, \vec{p}) \Phi_j D(p^0, \vec{p}) \frac{2p^0 e^{-ip^0T+i\vec{p}\vec{L}}}{p^2 - m_j^2 + i\epsilon}
\]

Integrate first over \( \vec{p} \), then over \( p^0 \equiv E \). Make use of Grimus-Stockinger theorem: for a large \( L \), \( A > 0 \) and a sufficiently smooth function \( \psi(\vec{p}) \),

\[
\int d^3 p \frac{\psi(\vec{p}) e^{i\vec{p}\vec{L}}}{A - \vec{p}^2 + i\epsilon} = -\frac{2\pi^2}{L} \psi(\sqrt{A\vec{L}}) e^{i\sqrt{A}L} + \mathcal{O}(L^{-3/2}) \Rightarrow
\]

\[
iA_{\alpha\beta}(T, \vec{L}) = \frac{-i}{8\pi^2 L} \sum_j U_{\alpha j}^* U_{\beta j} \int dE \Phi_P(E, p_j \vec{l}) \Phi_D(E, p_j \vec{l}) 2E e^{-iET+i\vec{p}_j\vec{L}}
\]

where

\[
p_j \equiv \sqrt{E^2 - m_j^2}, \quad \vec{l} \equiv \frac{\vec{L}}{L},
\]
Introduce

\[ \tilde{P}_{\alpha\beta}^{\text{tot}}(\vec{L}) = \int dT P_{\alpha\beta}(T, \vec{L}) = \frac{1}{8\pi^2} \frac{1}{4\pi L^2} \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k} \]

\[ \times \int dE \Phi_P(E, p_j \vec{l}) \Phi_D(E, p_j \vec{l}) \Phi_P^*(E, p_k \vec{l}) \Phi_D^*(E, p_k \vec{l}) (2E)^2 e^{i(p_j - p_k)L} \]

Neutrino production probability:

\[ P_{\alpha}^{\text{prod}} = \sum_j |U_{\alpha j}|^2 \int \frac{d^3 p_j}{(2\pi)^3} |\Phi_p(E, p_j)|^2 = \sum_j |U_{\alpha j}|^2 \frac{1}{8\pi^2} \int dE |\Phi_p(E, p_j)|^2 4E p_j \]

Detection probability:

\[ P_{\beta}^{\text{det}}(E) = \sum_k |U_{\beta k}|^2 |\Phi_D(E, p_k)|^2 \frac{1}{V}, \]
Oscillation probability in QFT

Let the number of particles $P_i$ entering the production region during time interval $T_0$ be $N_P$ and number of $D_i$ entering the detection region be $N_D$. Probability of neutrino emission during the finite interval of time $t$:

$$P_{\alpha}^{\text{prod}}(t) = N_P \int_0^t \frac{dt_P}{T_0} P_{\alpha}^{\text{prod}} = N_P P_{\alpha}^{\text{prod}} \frac{t}{T_0}, \quad \text{rate:} \quad \Gamma_{\alpha}^{\text{prod}} = N_P P_{\alpha}^{\text{prod}} \frac{1}{T_0}$$

Detection cross section:

$$\sigma_{\beta}(E) = \frac{N_D}{T_0} \sum_k |U_{\beta k}|^2 |\Phi_{kD}(E)|^2 \frac{E}{p_k}$$

Probability of the overall production-propagation-detection process:

$$P_{\alpha\beta}^{\text{tot}}(t, L) = \frac{N_P N_D}{T_0^2} \int_0^t dt_D \int_0^t dt_P P_{\alpha\beta}^{\text{tot}}(T, L) \Rightarrow$$
New integration variables $\tilde{T} \equiv (t_P + t_D)/2$ and $T = t_D - t_P \Rightarrow$

\[
P_{\alpha\beta}^{\text{tot}}(t, L) = \frac{N_P N_D}{T_0^2} \left[ \int_0^t dT P_{\alpha\beta}^{\text{tot}}(T, L)(t - T) + \int_0^t dT P_{\alpha\beta}^{\text{tot}}(T, L)(t + T) \right]
\]

\[
= \frac{N_P N_D}{T_0^2} \left[ t \int_{-t}^t dT P_{\alpha\beta}^{\text{tot}}(T, L) - \int_0^t dT T P_{\alpha\beta}^{\text{tot}}(T, L) + \int_{-t}^t dT T P_{\alpha\beta}^{\text{tot}}(T, L) \right]
\]

\[
\equiv \frac{N_P N_D}{T_0^2} \left[ tI_1(t) - I_2(t) + I_3(t) \right].
\]

For large $t$ (much larger than the time scales of the neutrino production and detection processes) $I_1 = \tilde{P}_{\alpha\beta}^{\text{tot}}(L)$ whereas $I_2 = I_3 = 0 \Rightarrow$

\[
P_{\alpha\beta}^{\text{tot}}(t, L) = \frac{N_P N_D}{T_0^2} t \tilde{P}_{\alpha\beta}^{\text{tot}}(L), \quad \Gamma_{\alpha\beta}^{\text{tot}}(L) = \frac{N_P N_D}{T_0^2} \tilde{P}_{\alpha\beta}^{\text{tot}}
\]
“$P_{\alpha\beta}(L, E)$” = \[
\frac{\sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k}^* U_{\beta k} \Phi_P(E, p_j) \Phi_D(E, p_k) \Phi_P^*(E, p_k) \Phi_D^*(E, p_k) e^{i(p_j - p_k)L} \sum_j |U_{\alpha j}|^2 |\Phi_P(E, p_j)|^2 p_j \sum_k |U_{\beta k}|^2 |\Phi_D(E, p_k)|^2 p_k^{-1}}{\sum_j |U_{\alpha j}|^2 |\Phi_P(E, p_j)|^2 p_j \sum_k |U_{\beta k}|^2 |\Phi_D(E, p_k)|^2 p_k^{-1}}
\]

For $|p_j - p_k| \ll p_j, p_k$ (ultra-relativistic or quasi-degenerate in mass $\nu$’s) and

\[
|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}
\]

one can replace

\[
p_j \rightarrow p, \quad \Phi_P(E, p_j) \rightarrow \Phi_P(E, p)
\]

$p - \text{average momentum}$

$\Rightarrow$ in the denominator of “$P_{\alpha\beta}(L, E)$”:

\[
\sum_j |U_{\alpha j}|^2 |\Phi_P(E, p_j)|^2 p_j \rightarrow |\Phi_P(E, p)|^2 p \sum_j |U_{\alpha j}|^2 = |\Phi_P(E, p)|^2 p,
\]

\[
\sum_k |U_{\beta k}|^2 |\Phi_D(E, p_k)|^2 p^{-1}_k \rightarrow |\Phi_D(E, p)|^2 p^{-1} \sum_k |U_{\beta k}|^2 = |\Phi_D(E, p)|^2 p^{-1},
\]
In the numerator of “$P_{\alpha\beta}(L, E)$” $\Phi_P$, $\Phi_D$ can be pulled out of the sums and canceled with those in the denominator. ⇒ stand. osc. probability:

$$P_{\alpha\beta}(L, E) = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* e^{-i\frac{\Delta m_{jk}^2}{2p} L}$$

Automatically satisfies unitarity, i.e. is properly normalized.

For $|p_j - p_k| \gg \sigma_p$ ($\Leftarrow \Delta m_{jk}^2/(2p) \gg \sigma_p$) – interf. terms strongly suppressed ⇒ Decoherence
The condition for the existence of well-defined oscillation probabilities is that neutrinos are either ultra-relativistic or nearly degenerate in mass and, in addition, the coherence prod./detection conditions are satisfied:

$$|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}$$

The QFT-based consideration clarifies the QM wave packet normalization prescription. QM and QFT approaches can be matched if the QM quantities $f_{jP}$ and $f_{jD}$ are identified with the QFT functions $\Phi_{jP}(E_j, \vec{p})$ and $\Phi_{jD}^*(E_j, \vec{p})$, respectively. But: the latter bear information not only on the properties of the emitted and absorbed neutrinos, but also on the production and detection processes. The QM normalization procedure is equivalent, in the limit $|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}$, to the division of the overall rate of the process by the production rate and detection cross section, as in QFT approach.
Coherence of $\nu$ production in different points

Neutrino production in extended sources: Amplitudes of neutrino emission in different points must be summed – a consistent QM procedure.

The standard approach: calculate the probability that neutrino produced at a fixed point $x$ oscillates, and then integrate over all $x$ in the source (probability summation procedure – classical in nature).

Both procedures give identical answers under realistic conditions!

The two approaches lead to different results whenever the localization properties of the parent particles at neutrino production and of the detection process are such that they prevent the precise localization of the point of neutrino emission – difficult to realize in practice.
Graphical interpretation

(a) Neutrino detector at $A$
(b) Neutrino detector at $A$, with muon detector at $\mu$
(c) Neutrino detector at $A$, with muon detector at $\mu$
(d) Neutrino detector at $A$, with muon detector at $\mu$
(e) Neutrino detector at $A$, with muon detector at $\mu$
(f) Neutrino detector at $A$, with muon detector at $\mu$
Additional phase for the segment $AB$:

$$\Delta \phi = -[E_j(P_j) - E_k(P_k)] \Delta t + (P_j - P_k) \Delta x.$$ 

$\Delta t$ and $\Delta x$: projections of $AB$ on the $t$ and $x$ axes. \Rightarrow

$$\Delta t = \frac{\sigma x}{v_g - v_\pi}, \quad \Delta x = \sigma x \frac{v_g}{v_g - v_\pi}.$$ 

$$\Delta \phi \simeq - \frac{v_g}{v_g - v_\pi} \cdot \frac{\Delta m^2_{jk}}{2P} \sigma x.$$
Are deviations between the results of the coherent amplitude summation and incoherent probability summation approaches experimentally observable? Requires extremely high energies of the parent pion:

$$2 \left( E_{\pi} \sigma_{x\pi} \right) \frac{\Delta m^2}{m_{\pi}^2} \gtrsim 1.$$ 

E.g. for $\sigma_{x\pi} \sim 10^{-4}$ cm and $\Delta m^2 \sim 1$ eV$^2$ $\Delta \phi$ would be $\sim 1$ for pion energies $E_{\pi} \gtrsim 10^3$ TeV – not feasible,

Another possibility: increase significantly the spatial width of w. packets of ancestor protons, which would increase the values of $\sigma_{x\pi}$. But: not clear how this could be achieved.

Other possibilities...
QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
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- Automatically produces correctly normalized oscillation probability and clarifies the normalization prescription of QM approach
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⇒ the simplistic QM wave packet approach may need QFT-motivated modifications; however, once they have been done, one can still work within the QM framework without losing any essential physics content.
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Yes!
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The standard formula for oscillation probability is stubbornly robust.
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Validity conditions:
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But: Conditions for partial decoherence are difficult to realize.
Problems with the plane-wave approach

- Same momentum $\Rightarrow$ oscillation probabilities depend only on time. Leads to a paradoxical result – no need for a far detector! “Time-to-space conversion” ($x = vt \approx t$) – assumes neutrinos to be point-like particles (notion opposite to plane waves).

- Same energy – oscillation probabilities depend only on coordinate. Does not explain how neutrinos are produced and detected at certain times. Corresponds to a stationary situation.

Plane wave approach $\Leftrightarrow$ exact energy-momentum conservation. Neutrino energy and momentum are fully determined by those of external particles $\Rightarrow$ only one mass eigenstate can be emitted!
QM uncertainty relations
Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations.

Neutrino energy and momentum are characterized by uncertainties $\sigma_E$ and $\sigma_p$ related to the spatial localization and time scale of the production and detection processes. These uncertainties:

- allow the emitted/absorbed neutrino state to be a coherent superposition of different mass eigenstates (Kayser, 1981)
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation (Nussinov, 1976)

$\sigma_E$ – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for $\sigma_p$. 
The paradox of $\sigma_E$ and $\sigma_p$

QM uncertainty relations: $\sigma_p$ is related to the spatial localization of the production (detection) process, while $\sigma_E$ to its time scale $\Rightarrow$ independent quantities.

On the other hand: Neutrinos propagating macroscopic distances are on the mass shell. For on-shell mass eigenstates $E^2 = p^2 + m_i^2$ means

$$E \sigma_E = p \sigma_p$$

How can this be understood?

The solution: At production, neutrinos are not on the mass shell. They go on shell only after they propagate $x \sim (a \text{ few}) \times$ De Broglie wavelengths. After that their energy and momentum get related by $E^2 = p^2 + m_i^2 \Rightarrow$ the larger uncertainty shrinks towards the smaller one to satisfy $E \sigma_E = p \sigma_p$.

On-shell relation between $E$ and $p$ allows to determine the less certain of the two through the more certain one, reducing the error of the former.
What determines the length of $\nu$ w. packets?

The length of $\nu$ w. packets: $\sigma_x \sim 1/\sigma_p$. For propagating on-shell neutrinos:

$$\sigma_p \simeq \min\{\sigma_p^{\text{prod}}, (E/p)\sigma_E^{\text{prod}}\} = \min\{\sigma_p^{\text{prod}}, (1/v_g)\sigma_E^{\text{prod}}\}$$

Which uncertainty is smaller at production, $\sigma_p^{\text{prod}}$ or $\sigma_E^{\text{prod}}$?

Consider neutrino production in decays of an unstable particle localized in a box of size $L_S$. Time between two collisions with the walls of the box: $T_S$.

- If $T_S < \tau$ ($\tau$ – lifetime of the parent unstable particle) $\Rightarrow$ $\sigma_E \simeq T_S^{-1}$ (collisional broadening). Mom. uncertainty: $\sigma_p \simeq L_S^{-1}$.
  
  But: $L_S = v_S T_S \Rightarrow \sigma_E < \sigma_p$ (a consequence of $v_S < 1$)

- If $T_S > \tau$ (quasi-free parent particle) $\Rightarrow$ $\sigma_E \simeq \tau^{-1} = \Gamma$.
  
  $\sigma_p \simeq [(p/E)\tau]^{-1} \simeq [(p/E)\sigma_E]^{-1}$, i.e. $\sigma_E \simeq (p/E)\sigma_p < \sigma_p$.
In both cases \( \sigma^\text{prod}_E < \sigma^\text{prod}_p \) \( \iff \) also when \( \nu \)'s are produced in collisions.

\[
\Rightarrow \quad \sigma_{\text{p eff}} \simeq \frac{\sigma_E}{v_g},
\]

\[
\sigma_x \simeq \frac{v_g}{\sigma_E}
\]

In the stationary limit \( (\sigma_E \to 0) \) one has \( \sigma_{\text{p eff}} \to 0 \) even though \( \sigma_p \) is finite! Therefore \( \sigma_x \to \infty \) and so the coherence length \( l_{\text{coh}} \to \infty \) – a well known result.
Coherence issues
Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations

Neutrino energy and momentum are characterized by uncertainties $\sigma_E$ and $\sigma_p$ related to the spatial localization and time scale of the production and detection processes. These uncertainties

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$\sigma_E$ – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for $\sigma_p$. 
When are neutrino oscillations observable?

Keyword: **Coherence**

Neutrino flavour eigenstates $\nu_e$, $\nu_\mu$ and $\nu_\tau$ are coherent superpositions of mass eigenstates $\nu_1$, $\nu_2$ and $\nu_3 \Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate $E$ and $p$ measurements one can tell (through $E = \sqrt{p^2 + m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear!

Full analogy with electron interference in double slit experiments: if one can establish which slit the detected electron has passed through, the interference fringes are washed out.
When are neutrino oscillations observable?

Another source of decoherence: wave packet separation due to the difference of group velocities $\Delta v$ of different mass eigenstates.

If coherence is lost: Flavour transition can still occur, but in a non-oscillatory way. E.g. for $\pi \rightarrow \mu \nu_i$ decay with a subsequent detection of $\nu_i$ with the emission of $e$:

$$P \propto \sum_i P_{\text{prod}}(\mu, \nu_i) P_{\text{det}}(e, \nu_i) \propto \sum_i |U_{\mu i}|^2 |U_{e i}|^2$$

– the same result as for averaged oscillations.

How are the oscillations destroyed? Suppose by measuring momenta and energies of particles at neutrino production (or detection) we can determine its energy $E$ and momentum $p$ with uncertainties $\sigma_E$ and $\sigma_p$. From $E_i = \sqrt{p_i^2 + m_i^2}$:

$$\sigma_{m^2} = \left[ (2E \sigma_E)^2 + (2p \sigma_p)^2 \right]^{1/2}$$
When are neutrino oscillations observable?

If \( \sigma_{m^2} < \Delta m^2 = |m_i^2 - m_k^2| \) – one can tell which mass eigenstate is emitted. \( \sigma_{m^2} < \Delta m^2 \) implies \( 2p\sigma_p < \Delta m^2 \), or \( \sigma_p < \Delta m^2 / 2p \simeq l_{osc}^{-1} \).

But: To measure \( p \) with the accuracy \( \sigma_p \) one needs to measure the momenta of particles at production with (at least) the same accuracy \( \Rightarrow \) uncertainty of their coordinates (and the coordinate of \( \nu \) production point) will be

\[
\sigma_{x, \text{prod}} \gtrsim \sigma_p^{-1} \sim l_{osc}
\]

\( \Rightarrow \) Oscillations washed out. Similarly for neutrino detection.

Natural necessary condition for coherence (observability of oscillations):

\[
L_{\text{source}} \ll l_{osc} , \quad L_{\text{det}} \ll l_{osc}
\]

No averaging of oscillations in the source and detector

Satisfied with very large margins in most cases of practical interest
Coherence production conditions:

\[ |\Delta E| \ll \sigma_E, \quad |\Delta p| \ll \sigma_p. \]

On the other hand:

\[ \Delta E \simeq v_g \Delta p + \frac{\Delta m^2}{2E}. \]

Constraint \( |\Delta E| \ll \sigma_E \) \Rightarrow

\[ \left| \frac{v_g \Delta p}{\sigma_E} + \frac{\Delta m^2}{2E \sigma_E} \right| \ll 1. \quad (\ast) \]

(a) The two terms in \( \Delta E \) do not approximately cancel each other. \( \Rightarrow \)
\[ v_g |\Delta p| \ll \sigma_E \leq \sigma_p, \text{ i.e. for relativistic neutrinos } |\Delta p| \ll \sigma_p \text{ follows from } |\Delta E| \ll \sigma_E. \]

(b1) There is a strong cancellation, but both terms on the l.h.s. of \( (\ast) \) are small – see case (a).

(b2) Strong cancellation, but both terms on the l.h.s. of \( (\ast) \) are \( \gtrsim 1 \): momentum condition is independent. But: the only known case – Mössbauer neutrinos.
Wave packet separation

Wave packets representing different mass eigenstate components have different group velocities $v_{gi}$ ⇒ after time $t_{coh}$ (coherence time) they separate ⇒ Neutrinos stop oscillating! (Only averaged effect observable).

Coherence time and length:

$$\Delta v \cdot t_{coh} \simeq \sigma_x ; \quad l_{coh} \simeq v t_{coh}$$

$$\Delta v = \frac{p_i}{E_i} - \frac{p_k}{E_k} \simeq \frac{\Delta m^2}{2E^2}$$

$$l_{coh} \simeq \frac{2E^2}{\Delta m^2} v \sigma_x$$

The standard formula for $P_{osc}$ completely neglects decoherence effects. How should it be modified when decoherence is present?
A manifestation of neutrino coherence

Even non-observation of neutrino oscillations at distances $L \ll l_{\text{osc}}$ is a consequence of and an evidence for coherence of neutrino emission and detection! Two-flavour example (e.g. for $\nu_e$ emission and detection):

$$A_{\text{prod/det}}(\nu_1) \sim U_{e1} = \cos \theta, \quad A_{\text{prod/det}}(\nu_2) \sim U_{e2} = \sin \theta$$

$$A(\nu_e \rightarrow \nu_e) = \sum_{i=1,2} A_{\text{prod}}(\nu_i) A_{\text{det}}(\nu_i) = \cos^2 \theta + e^{-i\Delta \phi} \sin^2 \theta$$

Phase difference $\Delta \phi$ vanishes at short $L$

$$P(\nu_e \rightarrow \nu_e) = (\cos^2 \theta + \sin^2 \theta)^2 = 1$$

If $\nu_1$ and $\nu_2$ were emitted and absorbed incoherently)

$$P(\nu_e \rightarrow \nu_e) \sim \sum_{i=1,2} |A_{\text{prod}}(\nu_i) A_{\text{det}}(\nu_i)|^2 \sim \cos^4 \theta + \sin^4 \theta < 1$$
Q.: When are the oscillations described by a universal (production and detection independent) oscillation probability?

A.: When neutrinos are relativistic or quasi-degenerate in mass and the conditions of coherent neutrino emission and detection

\[ \Delta E \ll \sigma_E, \quad \Delta p \ll \sigma_p \]

are satisfied.

Under these conditions the rate of the overall neutrino production-propagation-detection process can be factorized into the production rate \( d\Gamma_{\alpha}^{\text{prod}}(E)/dE \), propagation (oscillation) probability \( P_{\alpha\beta}(E, L) \) and detection cross section \( \sigma_{\beta}(E) \Rightarrow P_{\alpha\beta}(E, L) \) can be extracted.
Lorentz invariance issues
1. “Paradox” of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \rightarrow \mu \nu_\mu$):

\[ \sigma_E \simeq \tau^{-1} = \Gamma_\pi, \quad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \left( = v_g \tau \right) \]
Lorentz invariance of oscillation probability

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For decay in flight: $\Gamma'_\pi = \left(\frac{m_\pi}{E_\pi}\right)\Gamma_\pi$. One might expect

$$\sigma'_x \simeq \frac{E_\pi}{m_\pi} \sigma_x > \sigma_x.$$
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On the other hand, if the decaying pion is boosted in the direction of the neutrino momentum, the neutrino w. packet should be Lorentz-contracted!
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On the other hand, if the decaying pion is boosted in the direction of the neutrino momentum, the neutrino w. packet should be Lorentz-contracted!

**The solution:** pion decay takes finite time. During the decay time the pion moves over distance \( l = u\tau' \) (“chases” the neutrino if \( u > 0 \)).

\[
\sigma'_x \simeq \frac{v_g'}{\Gamma'} - l = v_g' \tau' - u\tau' = (v_g' - u)\gamma_u \tau = \frac{v_g \tau}{\gamma_u (1 + v_g u)},
\]

[the relativ. law of addition of velocities: \( v_g' = (v_g + u)/(1 + v_g u) \)].
Lorentz invariance issues – contd.

That is

\[
\sigma'_x = \frac{\sigma_x}{\gamma_u (1 + v_g u)}
\]

For relativistic neutrinos \( v_g \approx v'_g \approx 1 \) \( \Rightarrow \)

\[
\sigma'_x = \sigma_x \sqrt{\frac{1 - u}{1 + u}}
\]

\( \Rightarrow \) when the pion is boosted in the direction of neutrino emission \((u > 0)\) the neutrino wave packet gets contracted; when it is boosted in the opposite direction \((u < 0)\) – the wave packet gets dilated.
The oscillation probability must be Lorentz invariant! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.
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How can we see Lorentz invariance of the standard formula for the oscillation probability? $P_{ab}$ depends on $L/p$ (contains factors $\exp[-i \frac{\Delta m^2_{ik}}{2p} L]$). Is $L/p$ Lorentz invariant?
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Lorentz transformations:

\[
\begin{align*}
L' &= \gamma_u (L + ut), \\
t' &= \gamma_u (t + uL), \\
E' &= \gamma_u (E + up), \\
p' &= \gamma_u (p + uE).
\end{align*}
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The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{osc}$. $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$. $\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$. 
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Lorentz invariance issues – contd.

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How can we see Lorentz invariance of the standard formula for the oscillation probability? $P_{ab}$ depends on $L/p$ (contains factors $\exp[-i\frac{\Delta m^2_{12}}{2p}L]$). Is $L/p$ Lorentz invariant? Lorentz transformations:

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The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{osc}$. $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$. $\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$. On the other hand: $v_g = p/E$ $\Rightarrow$ $p' = \gamma_u p(1 + u/v_g)$. $\Rightarrow$ $L'/p' = L/p$. 
A more general argument (applies also to Mössbauer neutrinos which are not pointlike): Consider the phase difference

\[ \Delta \phi = -\frac{1}{v_g}(L - v_g t)\Delta E + \frac{\Delta m^2}{2p} L \]

- a Lorentz invariant quantity, though the two terms are in not in general separately Lorentz invariant.

**But:** If the 1st term is negligible in all Lorentz frames, the second term is Lorentz invariant by itself \( \Rightarrow L/p \) is Lorentz invariant.

The 1st term can be neglected when the production/detection coherence conditions are satisfied. In particular, it vanishes in the limit of pointlike neutrinos \( L = v_g t \). N.B.:

\[ L' - v_g' t' = \gamma_u \left[ (L + ut) - \frac{v_g + u}{1 + v_g u} (t + uL) \right] = \frac{L - v_g t}{\gamma_u (1 + v_g u)}, \]

i.e. the condition \( L = v_g t \) is Lorentz invariant. MB neutrinos: \( \Delta E \approx 0 \).
The oscillation probability must be Lorentz invariant even when the coherence conditions are not satisfied!

Lorentz invariance is enforced by the normalization condition.

\[ P_{ab}(L) = \sum_{i,k} U_{ai} U_{bi}^* U_{ak}^* U_{bk} I_{ik}(L), \quad \text{where} \]

\[ I_{ik}(L) \equiv \int dt \, A_i(L,t) A_k^*(L,t) e^{-i\Delta \phi_{ik}} \]

From the norm. cond. \( \int dt |A_i(L,t)|^2 = 1 \) \( \Rightarrow \)

\[ |A_i|^2 dt = \text{inv.} \quad \Rightarrow \quad |A_i||A_k| dt = \text{inv.} \quad \Rightarrow \quad A_i A_k^* dt = \text{inv.} \]

The phase difference \( \Delta \phi_{ik} = \Delta E_{ik} t - \Delta p_{ik} L \) is also Lorentz invariant \( \Rightarrow \)
so is \( I_{ik}(L) \), and consequently \( P_{ab}(L) \).
Coherence of $\nu$ production in different points
Calculating the oscillation probabilities

The transition amplitude:

$$A_{\alpha\beta}(L, t) = \sum_j U_{\alpha j}^* U_{\beta j} A_j(L, t)$$

Contribution of $\nu_j$:

$$A_j(L, t) \equiv \int dx \psi_j^D(x) \psi_j^S(x, t).$$

The detected state $\nu_j^D(x)$: a wave packet centered on the point $x = L$.

Assume the detection process is well localized:

$$\psi_j^D(x, t) = \delta(x - L) \Rightarrow$$

$$A_j(L, t) = \psi_j^S(L, t).$$

The oscillation probability:

$$P_{\alpha\beta}(L) = \sum_{j, k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* I_{jk}(L),$$
Here:

\[ I_{jk}(L) \equiv \int_{-\infty}^{\infty} dt \, A_j(L,t) A^*_k(L,t) = \int_{-\infty}^{\infty} dt \, \psi_j^S(L,t) \psi_k^{S*}(L,t). \]

\[ \downarrow \]

\[ I_{jk}(L) = \frac{1}{(1 - e^{-\Gamma l_p/v_\pi})} \cdot \frac{i \Gamma}{v_\pi \Delta m^2_{jk} / 2P} + i \Gamma \left[ e^{-i \frac{\Delta m^2_{jk}}{2P}} L - e^{-\Gamma l_p / v_\pi} e^{-i \frac{\Delta m^2_{jk}}{2P}} (L-l_p) \right] \]

The absolute normalization fixed by imposing the unitarity constraint

\[ \sum_\beta P_{\alpha \beta}(L) = 1 \quad \Rightarrow \quad I_{jj}(L) = 1. \]

Consider SBL experiments in the 3+1 scheme (only \( \Delta m^2_{41} \equiv \Delta m^2 \) can be considered to be nonzero) \( \Rightarrow \) an effective two-flavour approximation

Survival probabilities \( P_{\alpha \alpha} : \ s = |U_{\alpha 4}|, \ c = (1 - |U_{\alpha 4}|^2)^{1/2}. \)

Transition probability \( P_{\alpha \beta} : \ \sin^2 2\theta = 4|U_{\alpha 4}|^2 |U_{\beta 4}|^2. \)
\[ P_{\mu\mu} = c^4 + s^4 + \frac{2c^2s^2}{\xi^2 + 1} \frac{1}{(1 - e^{-\Gamma l_p/v_\pi})} \left[ \cos \phi + \xi \sin \phi \right] \]

\[ -e^{-\Gamma l_p/v_\pi} \left[ \cos(\phi - \phi_p) + \xi \sin(\phi - \phi_p) \right] \]

Here:

\[ \phi \equiv \frac{\Delta m^2}{2P} L, \quad \phi_p \equiv \frac{\Delta m^2}{2P} l_p, \quad \xi \equiv v_\pi \frac{\Delta m^2}{2P \Gamma}, \]
\[ P_{\mu\mu} = c^4 + s^4 + \frac{2c^2s^2}{\xi^2 + 1} \frac{1}{1 - e^{-\Gamma l_p/v_\pi}} [\cos \phi + \xi \sin \phi \]

\[ -e^{-\Gamma l_p/v_\pi} [\cos(\phi - \phi_p) + \xi \sin(\phi - \phi_p)] \]

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N.B.: for non-zero \( \xi \) and \( \Gamma l_p/v_\pi \) the probability \( P_{\mu\mu} \neq 1 \) even for \( L = 0 \) – “zero-distance” effect, a consequence of production coherence violation.
\( \nu_\mu \) survival probability

\[
P_{\mu\mu} = c^4 + s^4 + \frac{2c^2s^2}{\xi^2 + 1} \frac{1}{(1 - e^{-\Gamma l_p/v_\pi})} \left[ \cos \phi + \xi \sin \phi - e^{-\Gamma l_p/v_\pi} [\cos (\phi - \phi_p) + \xi \sin (\phi - \phi_p)] \right]
\]

Here:

\[
\phi \equiv \frac{\Delta m^2}{2P} L, \quad \phi_p \equiv \frac{\Delta m^2}{2P} l_p, \quad \xi \equiv v_\pi \frac{\Delta m^2}{2P \Gamma},
\]

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The parameter \( \xi \): essentially the ratio of \( \Delta m^2/2P \) and \( \Gamma \) – what we expected to be the (de)coherence parameter. For \( \xi \gg 1 \) the oscillating term strongly suppressed due to production decoherence (unless \( \Gamma l_p/v_p \ll 1 \)).
A relation between $\xi$, $\phi_p$ and $\Gamma_{lp}/v_\pi$:

$$\xi \cdot \frac{\Gamma_{lp}}{v_\pi} = \phi_p.$$ 

In the limit $\Gamma_{lp}/v_p \ll 1$:

$$P_{\mu\mu} = c^4 + s^4 + \frac{2s^2c^2}{\phi_p} \left[ \sin \phi - \sin(\phi - \phi_p) \right].$$

The decoherence parameter is $\phi_p$. For $\phi_p \gg 1$: $P_{\mu\mu} \approx \bar{P}_{\mu\mu} = c^4 + s^4$.

For $\phi_p \ll 1$:

$$P_{\mu\mu} = P^{\text{stand}}_{\mu\mu} = c^4 + s^4 + 2s^2c^2 \cos \phi.$$

This result does not depend on whether $\xi$ is small or large!

Two distinct regimes:

- $\Gamma_{lp}/v_\pi \ll 1$ (pion decay length large compared to $l_p$, decoherence parameter: $\phi_p$)
- $\Gamma_{lp}/v_\pi \gg 1$ (pion decay length small compared to $l_p$, decoherence parameter: $\xi$.)
How can this be understood?

The prod. coherence condition (ensures that different neutrino mass eigenstates forming a flavor neutrino state are emitted coherently):

$$\Delta E \ll \sigma_E,$$

For a decay of a free particle in a box (e.g. decay tunnel):

$$\sigma_E \sim \max\{\Gamma, v_\pi/l_p\} \cdot \frac{v_g}{v_g - v_\pi}.$$

$\sigma_E$ also determines the spatial width of the neutrino wave packet: $\sigma_x \sim v_g/\sigma_E$

$\Rightarrow$ for $\Gamma l_p/v_\pi \gg 1$ the width of the wave packet $\sigma_x \sim (v_g - v_\pi)/\Gamma$;

for $\Gamma l_p/v_\pi \ll 1$ $\sigma_x \sim [(v_g - v_\pi)/v_\pi] l_p$.

$$\Delta E \equiv |E_j(P_j) - E_k(P_k)| \sim \frac{\Delta m^2}{2P} \frac{v_\pi v_g}{(v_g - v_\pi)} \Rightarrow$$
The decoherence parameters

Two limiting regimes:

- \( \Gamma l_p/v_\pi \ll 1 \) (pion decay length \( l_{\text{decay}} = v_\pi/\Gamma \gg l_p \))

\[
\frac{\Delta E}{\sigma_E} \simeq \frac{\Delta m^2}{2P} l_p = \phi_p.
\]

- \( \Gamma l_p/v_\pi \gg 1 \) (pion decay length \( l_{\text{decay}} = v_\pi/\Gamma \ll l_p \))

\[
\frac{\Delta E}{\sigma_E} \simeq \frac{\Delta m^2}{2P} \cdot \frac{v_\pi}{\Gamma} = \xi.
\]

When expressed through \( l_{\text{osc}} = 4\pi E/\Delta m^2 \):

\[
\phi_p = 2\pi \frac{l_p}{l_{\text{osc}}}, \quad \xi = 2\pi \frac{l_{\text{decay}}}{l_{\text{osc}}}.
\]

The meaning of the production coherence condition:

The osc. phase acquired over the neutrino production region must be small \( \iff \) the production region must be small compared to \( l_{\text{osc}}/2\pi \).
The decoherence parameters – contd.

For $\Gamma l_p/v_\pi < 1$ ($l_p < l_{\text{decay}}$) ⇒ $l_{\text{prod}} \sim l_p$

For $\Gamma l_p/v_\pi > 1$ ($l_p > l_{\text{decay}}$) ⇒ $l_{\text{prod}} \sim l_{\text{decay}}$ ($< l_p$).

The condition $l_p \ll l_{\text{osc}}/2\pi$ in any case guarantees that the production coherence condition is satisfied.

What was wrong with the argument that the prod. coherence always requires $\xi \ll 1$? $\xi = v_P(\Delta m^2/2P\Gamma)$ becomes $> 1$ for small enough $\Gamma$ (long-lived parent particle). But then $l_{\text{dec}}$ may exceed $l_p$, and prod. coherence will be governed by $\phi_p$ rather than by $\xi$.

$$\xi > 1 \Rightarrow \frac{\Gamma}{v_\pi} < \frac{\Delta m^2}{2P}; \quad l_p < l_{\text{dec}} \Rightarrow \frac{\Gamma}{v_\pi} < l_p^{-1}.$$  

The second cond. follows from the first if $\Delta m^2/2P < l_p^{-1}$, i.e. $\phi_p < 1$.

It is only in the case $\phi_p > 1$ ($\gg 1$) that the production coherence may be governed by $\xi$. 

Evgeny Akhmedov
A completely different approach

Assume each neutrino production event is completely coherent \( \Rightarrow \) neutrino flavor transitions are described by the standard oscillation formula.

E.g. for \( P_{\mu\mu} \):

\[
P_{\mu\mu}(E, L) = P_{\mu\mu}^{\text{stand}}(L, E) = c^4 + s^4 + 2s^2c^2 \cos \phi.
\]

Sum the effects of pion decays at different points along the decays tunnel at the probabilities level:

\[
F_{\mu}(E, L) = F_{\pi}(E, 0)\Gamma \int_0^{l_p} e^{-\frac{\Gamma}{v_{\pi}}x} P_{\mu\mu}^{\text{stand}}(E, L - x)dx
\]

The effective oscillation probability:

\[
P_{\mu\mu}^{\text{eff}}(L, E) \equiv \frac{F_{\mu}(L, E)}{F_{\mu}^0(L, E)}
\]

(Hernandez & Smirnov, 2011; also performed a simplified calculation at the amplitude level)
Effective probability

The result:

\[ P_{\mu\mu}^{\text{eff}} = c^4 + s^4 + \frac{2c^2 s^2}{\xi^2 + 1} \frac{1}{(1 - e^{-\Gamma_l/v\pi})} \left[ \cos \phi + \xi \sin \phi \right. \]

\[ -e^{-\Gamma_l/v\pi} \left[ \cos(\phi - \phi_p) + \xi \sin(\phi - \phi_p) \right] \]

Exactly the same expression as that obtained by summation (integration) at the amplitude level!
The result:

\[ P_{\mu\mu}^{\text{eff}} = c^4 + s^4 + \frac{2c^2 s^2}{\xi^2 + 1} \frac{1}{(1 - e^{-\Gamma l_p/v_\pi})} \left[ \cos \phi + \xi \sin \phi \right. \]

\[ \left. -e^{-\Gamma l_p/v_\pi} [\cos(\phi - \phi_p) + \xi \sin(\phi - \phi_p)] \right] \]

Exactly the same expression as that obtained by summation (integration) at the amplitude level!

The simple integration of the oscillation probability along the decay tunnel (source) – in general with the decay exponential taken into account – automatically takes care of the production coherence condition!

Prod. coherence ⇔ localization of the ν production process.
Why does this happen?

It is well known that decoherence gives the same result as averaging; but why the exact probabilities (with arbitrary degree of coherence) exactly coincide in the two cases?
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The spatial size of the production region is very small — given by the size of the smallest WP of the particles participating in neutrino production, in our case of the pion. Since we consider pions to be point-like and detection to be fully localized, the position of the production point is completely defined. ⇒

There is no interference between the amplitudes of neutrino production in different (even closely located) points.
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For non-zero $\sigma_{x\pi}$ one can expect some differences between the results of the two approaches, but for $\sigma_{x\pi} \ll \min\{l_p, l_{dec}\}$ they should be small (though some enhancement possible).
Two models of finite-size pion WP, Gaussian and box-type. For \( \Gamma l_p/v_\pi \gg 1 \):

\[
P_{\mu\mu}^{\text{eff}} = c^4 + s^4 + \frac{2c^2 s^2}{\xi^2 + 1} [(\cos \phi + \xi \sin \phi) - A_\pi \xi (\xi \cos \phi - \sin \phi)]
\]

The parameter \( A_\pi \):

\[
A_{\pi \text{box}} = \frac{v_g}{v_\pi} \frac{\Gamma}{v_g - v_\pi} \sigma_{x\pi}, \quad A_{\pi \text{Gauss}} = \frac{2}{\sqrt{2\pi}} \frac{v_g}{v_\pi} \frac{\Gamma}{v_g - v_\pi} \sigma_{x\pi}.
\]

i.e. \( A_\pi \sim (v_g/v_\pi) \sigma_{x\pi}/\sigma_{x\nu} \). The correction is of order

\[
A_\pi \xi \sim \left[ \frac{\Delta m^2}{2P} \sigma_{x\pi} \right] \cdot \frac{v_g}{v_g - v_\pi} = 2\pi \frac{\sigma_{x\pi}}{l_{\text{osc}}} \cdot \frac{v_g}{v_g - v_\pi}
\]

– small since \( \sigma_{x\pi} \ll l_{\text{osc}} \) (unless \( v_\pi \simeq v_g \) to a very high accuracy).

An interesting point: summation at the probabilities level for finite-thickness (= \( d \)) proton target and point-like neutrino production gives similar expression, but with \( A_\pi \xi = (\Delta m^2/2P)d \) (no factor \( [v_g/(v_g - v_\pi)] \)).
Effects of muon detection (for pointlike pion)

If muons is detected: plane wave $\rightarrow$ wave packet

$$\psi_\mu(x, t) = e^{iKx - iE_\mu(K)t} g_\mu[(x - x_S) - v_\mu(t - t_S)].$$

Shape factor \( g_\mu[(x - x_S) - v_\mu(t - t_S)] \) determined by the muon detection process. The argument of \( g_\mu \): initial condition that at time \( t = t_S \) the peak of the w. packet is at \( x = x_S \). Choose \( x_S \) as the coordinate of the center of the muon w. packet at the neutrino production time. For pointlike pions \( x_S \) should lie on the pion’s trajectory \( \Rightarrow x_S = v_\pi t_S \).

$$I_{jk}(L) = C_0 \int_0^{l_p} dx \left| g_\mu((x - x_S) \frac{v_\pi - v_\mu}{v_\pi}) \right|^2 e^{-i \frac{\Delta m^2}{2} \frac{2}{v_\pi^2} (L-x) - \Gamma \frac{x}{v_\pi}}.$$

When the muon is undetected: \( g_\mu \rightarrow 1 \). Eff. width of the muon w. packet:

$$\tilde{\sigma}_{x\mu} \equiv \sigma_{x\mu} \frac{v_\pi}{v_\pi - v_\mu}.$$

The results of amplitude summation and probability summation approaches again coincide.
Limiting cases

(1) $\tilde{\sigma}_{x\mu} \rightarrow \infty$: plane wave limit. $g_\mu \rightarrow const$ – previous results recovered.
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(2) $\tilde{\sigma}_{x\mu} \to 0$: pointlike muon limit, $g_\mu \propto \delta(x - x_S)$.

$$I_{jk}(L) = \text{const.} e^{-\Gamma \frac{x_S}{v_\pi}} e^{-i \frac{\Delta m^2_{jk}}{2\Gamma}(L - x_S)} \Rightarrow P_{\alpha\beta}^{\text{stand}}(L - x_S).$$

No production decoherence effects.
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\]

No production decoherence effects.

For \( \tilde{\sigma}_{x\mu} \not\rightarrow \infty \) – oscillations of a “tagged” neutrino, i.e. of a neutrino produced together with the muon which was detected and whose production coordinate was found to be \( x_S \) with the accuracy \( \tilde{\sigma}_{x\mu} \).
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$$I_{jk}(L) = \text{const.} e^{-\Gamma \frac{x_S}{v_\pi}} e^{-i \frac{\Delta m_{jk}^2}{2P}(L-x_S)} \Rightarrow P_{\alpha\beta}^{\text{stand}}(L-x_S).$$

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If $\tilde{\sigma}_{x\mu} \gg l_p$, previous results are recovered. For $\tilde{\sigma}_{x\mu} \ll \min\{l_p, x_S\}$ \Rightarrow

$$P_{\mu\mu} = c^4 + s^4 + 2s^2c^2 e^{-\frac{1}{2} \left(\frac{\Delta m_{jk}^2}{2P}\right)^2 \tilde{\sigma}_{x\mu}^2} \cos\left(\frac{\Delta m_{jk}^2}{2P}(L-x_S)\right).$$
Limiting cases

(1) \( \tilde{\sigma}_{x\mu} \to \infty \): plane wave limit. \( g_{\mu} \to \text{const} \) – previous results recovered.

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\[
I_{jk}(L) = \text{const}. e^{-\Gamma x_S/v_{\pi}} e^{-i \frac{\Delta m^2_{jk}}{2P}(L-x_S)} \Rightarrow P^{\text{stand}}_{\alpha\beta}(L-x_S).
\]

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If \( \tilde{\sigma}_{x\mu} \gg l_p \), previous results are recovered. For \( \tilde{\sigma}_{x\mu} \ll \text{min}\{l_p, x_S\} \) \( \Rightarrow \)

\[
P_{\mu\mu} = c^4 + s^4 + 2s^2c^2 e^{-\frac{1}{2} \left( \frac{\Delta m^2}{2P} \right)^2 \tilde{\sigma}_{x\mu}^2} \cos \left( \frac{\Delta m^2}{2P}(L-x_S) \right).
\]

\( \Rightarrow \) the decoherence parameter is \( \frac{\Delta m^2}{2P} \tilde{\sigma}_{x\mu} \). For \( \tilde{\sigma}_{x\mu} \ll l_{\text{osc}}/2\pi \) the stand. probability is recovered.
The case when the muon interacts with the medium but there are no muon detectors (the muon’s position not measured). Neutrinos are not tagged ⇒ one has to integrate

\[ I_{jk}(L) = C_0 \int_0^{l_p} dx |g_\mu((x - x_S) \frac{v_\pi - v_\mu}{v_\pi})|^2 e^{-i \frac{\Delta m^2_{jk}}{2P} (L-x) - \Gamma \frac{x}{v_\pi}}. \]

over \( x_S \).

Integration of \( |g_\mu|^2 \) gives the normalization constant of this function which does not influence the oscillation probabilities. The results obtained in the case when the muon is not detected are recovered.
Interaction of the pions in the bunch btw themselves or with other particles may identify the individual pion whose decay produces a given neutrino. E.g. pion decay may lead to some recoil of the neighbouring particles which may be detected. ⇒

Would localize the neutrino production point up to an uncertainty of order of the inter-pionic distance (or the distance between the pion and the other particles in the source) $r_0$ ⇒ neutrino tagging.

Production decoherence parameter: $\left(\Delta m^2 / 2P\right) r_0$.

If the information about the interaction between the decaying pion and the surrounding particles is not recorded and not used for neutrino tagging, the oscillations occur in exactly the same way as if pions did not interact with each other or with other particles.
Unless otherwise specified, $\Delta m^2 = 2 \text{ eV}^2$. For $\beta$-beams $E_0 = 2 \text{ MeV}$, $\tau_0 = 1\text{s}$, $\gamma = 100$.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\langle E_\nu \rangle$(MeV)</th>
<th>$L$(m)</th>
<th>$l_p$(m)</th>
<th>$l_{\text{dec}}$(m)</th>
<th>$l_{\text{osc}}$(m)</th>
<th>$\phi$</th>
<th>$\Gamma l_p/v_P$</th>
<th>$\phi_p$</th>
<th>$\xi$</th>
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<tbody>
<tr>
<td>LSND</td>
<td>$\sim$40</td>
<td>30</td>
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<td>-</td>
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<tr>
<td>NOMAD (20 eV$^2$)</td>
<td>2.7 $\cdot$ 10$^3$</td>
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<td>290</td>
<td>3009</td>
<td>33480</td>
<td>0.145</td>
<td>0.1</td>
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<tr>
<td>CCFR (10$^2$ eV$^2$)</td>
<td>5 $\cdot$ 10$^4$</td>
<td>891</td>
<td>352</td>
<td>5570</td>
<td>1240</td>
<td>4.51</td>
<td>0.06</td>
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<tr>
<td>CDHS (20 eV$^2$)</td>
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<td>52</td>
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<tr>
<td>$\beta$-beams</td>
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<td>1.3 $\cdot$ 10$^5$</td>
<td>2500</td>
<td>3 $\cdot$ 10$^{10}$</td>
<td>496</td>
<td>1647</td>
<td>8.3 $\cdot$ 10$^{-8}$</td>
<td>31.7</td>
<td>3.8 $\cdot$ 10$^8$</td>
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</tbody>
</table>

Noticeable effects for MiniBooNE, NOMAD (20 eV$^2$), CCFR (100 eV$^2$), CDHS (20 eV$^2$), K2K, T2K, MINOS, NO$\nu$A, very large effects for $\beta$-beams
Examples of prod. coherence violation

$\nu_e \rightarrow \nu_s$ oscillations in $\beta$-beam expts. (Agarwalla, Huber & Link, arXiv:0907.3145).

Ratio of oscillated and unoscillated fluxes ($\gamma = 30$, $l_p = 10$ m, $L = 50$ m):

![Graphs showing oscillations in fluxes for T2K and CCFR experiments](image-url)
Do charged leptons oscillate?
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What do we mean by charged leptons?

The usual $e^\pm$, $\mu^\pm$ and $\tau^\pm$ are mass eigenstates $\Rightarrow$ do not oscillate.
Can we imagine a situation when one creates a coherent superposition of $e$, $\mu$ and $\tau$ and then also detects their coherent superposition (the same or different) rather than individual mass-eigenstate charged leptons?

Charged - current weak interactions look completely symmetric w.r.t. neutrinos and charged leptons!

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} (\bar{e}_a L \gamma^\mu U_{a i} \nu_{i L}) W^-_\mu + \text{h.c.}, \quad U = V_l^\dagger V_\nu$$

Why do we say that charged leptons are emitted and detected in mass eigenstates and neutrinos in flavour states (superpositions of mass eigenstates) and not vice versa? Or not both as some superpositions of mass eigenstates?

E.g.

$$|e_1\rangle = U_{1 e} |e\rangle + U_{1 \mu} |\mu\rangle + U_{1 \tau} |\tau\rangle$$

is emitted or detected together with $\nu_1$,

$$|e_2\rangle = U_{2 e} |e\rangle + U_{2 \mu} |\mu\rangle + U_{2 \tau} |\tau\rangle$$

is emitted or detected together with $\nu_2$,

$$|e_3\rangle = U_{3 e} |e\rangle + U_{3 \mu} |\mu\rangle + U_{3 \tau} |\tau\rangle$$

is emitted or detected together with $\nu_3$. 
Why do neutrinos oscillate?

Because they are emitted (and absorbed) alongside charged leptons of definite mass $e^\pm$, $\mu^\pm$ or $\tau^\pm$. (This “measures” the flavour of neutrinos).

How do we know that charged leptons are in mass eigenstates?

(1) Beta decay: only electrons are emitted together with neutrinos. Emission of $\mu^\pm$ and $\tau^\pm$ is forbidden by energy conservation.

(2) Decays $\pi^\pm \to \mu^\pm \nu$, $\pi^\pm \to e^\pm \nu$ (or $K^\pm \to \mu^\pm \nu$, $K^\pm \to e^\pm \nu$). Here emission of both muons and electrons is allowed.

Assume a coherent superposition of $e$ and $\mu$ is produced in pion decay (nearly) at rest. The energy uncertainty of the charged lepton:

$$\sigma_E \simeq \Gamma_\pi = 2.5 \cdot 10^{-8} \text{ eV}$$

Uncertainty in the mass determination ($\sqrt{(2E\sigma_E)^2 + (2p\sigma_p)^2} \simeq 2\sqrt{2}E\sigma_E$):

$$\sigma_{m^2} \sim 2\sqrt{2}E\sigma_E \simeq 2\sqrt{2} \cdot (90 \text{ MeV}) \cdot (2.5 \cdot 10^{-8} \text{ eV}) \simeq 6.4 \text{ eV}^2$$
Do charged leptons oscillate?

This has to be compared with \( m_\mu^2 - m_e^2 \simeq (106 \text{ MeV})^2 \) \( \Rightarrow \) Different mass-eigenstate charged leptons are emitted incoherently!

This provides a “measurement” of the flavour of the emitted neutrino

For pion decay in flight: assume pion’s energy is \( E_0 \). The energies of the produced charged leptons are rescaled as \( E \rightarrow E (E_0/m_\pi) \), but the pion decay width (and so \( \sigma_E \)) is rescaled as \( \Gamma_\pi \rightarrow \Gamma_\pi (m_\pi/E_0) \) \( \Rightarrow \) \[ \left[ (2E\sigma_E)^2 + (2p\sigma_p)^2 \right]^{1/2} \] remains the same \( (\sigma_{m^2} \text{ a Lorentz invariant quantity}) \).

\[ \Downarrow \]

◊ Charged leptons produced in \( \pi^\pm \rightarrow l^\pm \nu \) and \( K^\pm \rightarrow l^\pm \nu \) decays are always emitted as mass eigenstates and not as coherent superpositions of different mass eigenstates because of their very large \( \Delta m^2 \).

◊ Therefore even oscillations between \( e_1, \ e_1 \) and \( e_3 \) (or any other superpositions of \( e, \mu \) and \( \tau \)) are not possible.
Do charged leptons oscillate?

The masses and decay widths of $\pi^\pm$, $K^\pm$ are rather small $\Rightarrow \sigma_{m^2}$ small. How about decays of $W^\pm$? For $W^\pm \rightarrow l^\pm \nu$ decays at rest:

$$\Gamma_{W \rightarrow l\nu}^0 \approx \frac{G_F m_W^3}{6\sqrt{2}\pi} \approx 230 \text{ MeV}$$

$$\Rightarrow \sigma_{m^2} \sim 2\sqrt{2} E \sigma_E \approx 2\sqrt{2} \cdot 40 \text{ GeV} \cdot 230 \text{ MeV} \approx (5 \text{ GeV})^2.$$  

Thus

$$\sigma_{m^2} \gg m_\mu^2 - m_e^2, \quad \sigma_{m^2} > m_\tau^2 - m_\mu^2 \approx (1.77 \text{ GeV})^2,$$

$$\Rightarrow \text{all three charged leptons are produced coherently in } W^\pm \text{ decays.}$$

Can one then observe oscillations between their different coher. superpositions? Coherence length $x_{coh} \approx \sigma_x / \Delta v_g$:

$$(x_{coh})_{\text{max}} \approx \left[ \Gamma_{W \rightarrow l\nu}(\Delta v_g)_{\text{min}} \right]^{-1} \approx \frac{3\sqrt{2}\pi}{G_F m_W (m_\mu^2 - m_e^2)} \approx 2.5 \times 10^{-8} \text{ cm}.$$  

$$\Rightarrow l^\pm \text{ loose their coherence almost immediately after their production}$$
Do charged leptons oscillate?

What about $W^\pm \rightarrow l^\pm \nu$ decays in flight? Let $\gamma$ be the Lorentz factor of $W^\pm$. 

$$(\Delta v_g)_{\text{min}} \simeq \Delta m^2_{\mu e}/2E^2 \equiv (m^2_\mu - m^2_e)/2E^2$$

and the partial decay width of $W^\pm$ scale with $\gamma$ as

$$(\Delta v_g)_{\text{min}} \rightarrow \gamma^{-2}(\Delta v_g)_{\text{min}}, \quad \Gamma^0_{W \rightarrow l_\alpha \nu} \rightarrow \gamma^{-1}\Gamma^0_{W \rightarrow l_\alpha \nu}.$$ 

Therefore the maximum coherence length

$$(l_{\text{coh}})_{\text{max}} \simeq \sigma_x/(\Delta v_g)_{\text{min}} \simeq 1/\left[\Gamma^0_{W \rightarrow l_\alpha \nu}(\Delta v_g)_{\text{min}}\right]$$

scales as

$$(l_{\text{coh}})_{\text{max}} \rightarrow \gamma^3(l_{\text{coh}})_{\text{max}}.$$ 

In order for $(l_{\text{coh}})_{\text{max}}$ to be larger than e.g. 1 m, one would need $\gamma \gtrsim 1600$, or $E_W \gtrsim 130$ TeV — far above presently feasible energies.
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N.B.: Even if coherence was satisfied for charged leptons, to fix the composition of the mixed $l^\pm$ state in terms of $e$, $\mu$ and $\tau$ one would have to detect the accompanying neutrino as a state different from $\nu_\text{fl}$ — e.g. as a mass eigenstate. Not possible within the standard model!
Consider the SM amended by three heavy RH neutrinos $N_i$ (seesaw model) plus an extra Higgs doublet. In this model $N_i$ can decay into a charged lepton and charged Higgs boson:

$$N_i \rightarrow e_i^- + \Phi^+.$$ 

Decays are caused by the Yukawa coupling Lagrangian

$$\mathcal{L}_Y = Y_{ai} \bar{L}_a N_{Ri} \Phi + h.c. ,$$

In the basis where the mass matrices of $N_i$ and $l^\pm$ have been diagonalized, the Yukawa coupling matrix $Y_{ai}$ is in general not diagonal $\Rightarrow$ in the decay of a mass-eigenstate sterile neutrino $N_i$ any of the three charged leptons $e_a = e, \mu, \tau$ can be produced.

What are the conditions for the produced charged lepton state $e_i$ to be a coherent superposition of the mass eigenstates $e_a$:

$$|e_i\rangle = [(Y^\dagger Y)_{ii}]^{-1/2} \sum_a Y_{ia}^\dagger |e_a\rangle ,$$

and how long this state can maintain its coherence?
Neglecting the masses of $\Phi^\pm$ and $l^\pm$ compared to the mass $M_i$ of the sterile neutrino:

$$\Gamma_i^0 \simeq \alpha_i M_i, \quad \text{where} \quad \alpha_i \equiv \frac{(Y^+Y)_{ii}}{16\pi}.$$ 

Coherent production condition:

$$2\sqrt{2} E \Gamma_i^0 \simeq 2\sqrt{2} \left( M_i/2 \right) \alpha_i M_i > \max\{m_\mu^2 - m_e^2, m_\tau^2 - m_\mu^2\},$$

or

$$\alpha_i > 2.2 \left( \text{GeV} / M_i \right)^2.$$ 

From $l_{coh} = \sigma_x v_g / \Delta v_g$ the coherence length for the emitted charged lepton state:

$$l_{coh} \simeq \frac{M_i^2}{2\Gamma_i^0 (m_\tau^2 - m_\mu^2)} \simeq 3.1 \times 10^{-15} \alpha_i^{-1} \frac{M_i}{\text{GeV}} \text{ cm}.$$

$\Rightarrow$
$l_{\text{coh}} < 1.4 \times 10^{-15} \text{ cm } (M_i/\text{GeV})^3$.

For $N_i$ decays in flight the r.h.s. has to be multiplied by $\gamma^3 \Rightarrow (M_i/\text{GeV})^3$ has to be replaced by $(E_i/\text{GeV})^3$.

The charged lepton state will maintain its coherence over the distance $\sim 1 \text{ m}$ if

$$E_i \gtrsim 400 \text{ TeV} \Rightarrow (Y^\dagger Y)_{ii} \gtrsim 1.3 \times 10^{-11}.$$  

If only $e$ and $\mu$ are to be produced coherently, a milder lower limit on $E_i$ results:

$$E_i \gtrsim 10 \text{ TeV} , \quad (Y^\dagger Y)_{ii} \gtrsim 8.5 \times 10^{-11}.$$
Extensions of the standard model?

If the condition for coherent creation of the charged lepton state is satisfied and this state is detected through the inverse decay process before it loses its coherence, it may exhibit oscillations: a mass eigenstate sterile neutrino $N_j$ different from $N_i$ can be produced in the detection process $\Rightarrow$ the state $e_i$ has oscillated into $e_j$.

Charged leptons would be able to oscillate, leading to a non-zero probability of the emission or absorption of a different sterile neutrino mass eigenstate $N_j$ in the processes $e_j^{\pm} + \Phi^{\mp} \rightarrow N_j$ or $e_j^{\pm} + N_j \rightarrow \Phi^{\pm}$.

$\Rightarrow$ The roles of neutrinos and charged leptons reversed compared to the usual situation because of sterile neutrinos being much heavier than the charged leptons.
Energy-mom. conservation and entanglement
How about energy-momentum conservation?

Conservation of energy and momentum is an exact law of nature.
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Rates of processes in quant. theory (gen. Fermi’s Golden rule):

\[ \Gamma = \prod_i \frac{1}{(2E_i)} \int \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f} |M_{fi}|^2 (2\pi)^4 \delta^4 \left( \sum_f p_f - \sum_i p_i \right) \]

The factor \( \delta^4 \left( \sum_f p_f - \sum_i p_i \right) \) ensures energy-momentum conservation.
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From 4-momenta of particles accompanying \( \nu \) production or detection one can find neutrino energy and momentum \( \Rightarrow \) through \( E^2 = p^2 + m^2 \) – the neutrino mass.
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The dichotomy led to a significant confusion in the literature.
How can it be resolved?
Possible solution: entanglement

Consider e.g. $\pi \rightarrow \mu + \nu$ decay.

Suppose that the 4-momentum of the pion $p_\pi$ is well defined but the muon 4-momentum is correlated with that of the emitted $\nu_i$:

$$p_{\nu i} + p_{\mu i} = p_\pi, \quad i = 1, 2, 3$$

State produced in the pion decay: a coherent superposition of different neutrino mass eigenstates accompanied by the muon states with correlated 4-momenta (entangled state):

$$|\mu \nu\rangle = \sum_i U_{\mu i}^* |\mu(p_{\mu i})\rangle |\nu_i(p_{\nu i})\rangle.$$ 

If muon 4-momentum is measured very accurately (e.g. $p_{\mu} = p_{\mu 1}$) $\Rightarrow$ neutrino detector should observe only $\nu_1$ with 4-momentum $p_{\nu 1}$.

A realization of the Einstein-Podolsky-Rosen correlation.

But: in this case no oscillations would occur!
⇒ Disentanglement is necessary.
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Assumed to be achieved through a measurement of the muon momentum with a sufficiently large intrinsic uncertainty (\(\Leftrightarrow\) sufficiently good localization of the measurement process). Leads to a violation of the strict correlation between the muon and neutrino 4-momenta \(\Rightarrow\) to a separation of the muon and neutrino parts of \(|\mu\nu\rangle\). Oscillations become possible.
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Entanglement – contd.

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$\diamond$ Kinematic entanglement is irrelevant to neutrino oscillations!
Mössbauer neutrinos
Mössbauer effect

Conventional Mössbauer effect – Res. absorption of $\gamma$ quanta:

$$A^* \rightarrow A + \gamma; \quad A + \gamma \rightarrow A^*$$

Nuclear exc. energy: $\omega_0$.
Recoil energy: $R = \frac{\omega_0^2}{2M}$

$$E_e = \omega_0 - \frac{\omega_0^2}{2M}$$

$$E_a = \omega_0 + \frac{\omega_0^2}{2M}$$

Recoilless emission and absorption (Mössb. eff.):

$$E_e \approx E_a \approx \omega_0$$

Strong enhancement of absorption
Beta decay with 2 - body final state:

\[ A(N, Z) \rightarrow A(N - 1, Z + 1) + e^-_B + \bar{\nu}_e \]

Inverse process:

\[ \bar{\nu}_e + e^-_B + A(N - 1, Z + 1) \rightarrow A(N, Z) \]

If the nuclei are embedded in solid state lattice, recoilless emission and absorption in principle possible.
Possibility of Mössbauer effect with neutrinos:
Visscher, 1959; Kells & Schiffer, 1983; Raghavan, 2005, 2006

Relevant processes considered:

Bahcall, 1961 – bound state $\beta$ decay;
Mikaelyan, Tsinoev & Borovoi, 1967 – inverse process
(stimulated K-electron capture)
Mössbauer effect with neutrinos on $^3\text{H} - ^3\text{He}$ system:

$$^3\text{H} \rightarrow (^3\text{He} + e^-_B) + \bar{\nu}_e; \quad \bar{\nu}_e + (^3\text{He} + e^-_B) \rightarrow ^3\text{H}$$

Energy release: $Q = 18.6 \text{ keV}$. Mean lifetime of $^3\text{H}$ is 17.8 yr $\Rightarrow$

Nat. linewidth $\Gamma_{^3\text{H}} = 1.17 \times 10^{-24} \text{ eV}$ – extremely small: $\Delta E/E \sim 10^{-28}$!
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Various (homogeneous and inhomogeneous) broadening effects exist. By suppressing them probably an effective linewidth $\Gamma_{\text{eff}} \sim 10^{-11}$ eV can be achieved (W. Potzel) $\Rightarrow$ $\Delta E/E \sim 10^{-15}$ – still very small.
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Number of $^3$H atoms produced in the target can be counted by detecting their decay or using mass spectroscopy.

Very serious technical difficulties exist, but apparently realization of a Mössbauer experiment with neutrinos is not impossible (Raghavan, Potzel).

If realized: for $\Gamma \sim 10^{-11} \text{ eV}, \quad \sigma \sim 10^{-33} \text{ cm}^2$!
Mössbauer effect with neutrinos?

If a Mössbauer neutrino experiment is realized ⇒ a unique source of extremely monochromatic low energy neutrinos. Would open up possibilities

- to detect for the first time keV neutrinos
- to detect neutrinos with g or 100 g scale (rather than t or kt scale) detectors
- to observe gravitational redshift of neutrinos
- to study neutrino oscillations at distances \( \sim 10 \text{ m} \) rather than km or hundreds/thousands of km
- to search for the effects of yet unmeasured mixing angle \( \theta_{13} \) and possibly measure it
- to discriminate between the normal and inverted neutrino mass hierarchies without using matter effects
- to study possible oscillations into sterile neutrino states
Will Mössbauer neutrinos oscillate?

Arguments in the literature (Bilenky et al.):

Mössbauer neutrinos may not oscillate because of their extremely small linewidth

(some energy uncertainty is usually necessary to ensure the coherence of flavour states)
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Is that true?
Will Mössbauer neutrinos oscillate?

Neutrino oscillations require some intrinsic uncertainty of energy and momentum of the emitted and detected neutrino states!

If $E$ and $p$ were known precisely, from $E^2 = p^2 + m_i^2$ one would determine which mass eigenstate has been emitted $\Rightarrow$ neutrinos of different mass would not be emitted coherently.

For Mössbauer effect with neutrinos in $^3\text{H} - ^3\text{He}$ system:

\[
\frac{\Delta m^2}{2E} = \frac{2.5 \times 10^{-3} \text{ eV}^2}{2 \cdot 18.6 \text{ keV}} \simeq 6.7 \cdot 10^{-8} \text{ eV} \gg \Gamma \sim 10^{-11} \text{ eV}!
\]

Can neutrinos of different mass be accommodated within such a small energy uncertainty?

Will neutrinos with such small energy uncertainty oscillate?
Problems with the plane wave approach

◊ Plane wave approach: plagued with inconsistencies. If applied correctly, does not lead to neutrino oscillations at all!
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Plane waves mean that energies and momenta of particles have sharp values

⇒ When applied to neutrino production and detection processes: neutrino $E$ and $\vec{p}$ can be determined from those of the accompanying particles.
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Neutrinos propagate macroscopic distances between their source and detector, i.e. are on the mass shell ⇒ their energy and momentum satisfy

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By knowing the neutrino energy and momentum one can determine its mass
But: Mass eigenstates do not oscillate!
Two “standard” approaches to \( \nu \) oscillations

The oscillation phase: \[ \phi = p_\mu x^\mu = E \cdot t - p \cdot x \Rightarrow \]

\[ \Delta \phi = \Delta E \cdot t - \Delta p \cdot L \]

I. Same momentum approach \((\Delta p = 0)\). The oscillation phase

\[ \Delta \phi = \Delta E \cdot t - \Delta p \cdot L \Rightarrow \Delta E \cdot t \]

- evolution in time; needs to use \( L \approx t \).

II. Same energy approach \((\Delta E = 0)\):

\[ \Delta \phi = - \Delta p \cdot L \]

- evolution in space.
– Same momentum approach (evolution in time): no.

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Our point of view: in general, there is no reason to believe that $\nu_i$ have either same energy or same momentum. No need to perform Mössbauer $\nu$ experiment to decide which approach is correct – it is sufficient to carefully examine the validity of the approximations used.
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Very small effective linewidth $\Gamma \Rightarrow$ small energy uncertainty of the emitted neutrino state. Can different neutrino mass eigenstates be emitted coherently?

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